Nonlinear Differential Equations in the Framework of Regular Variation

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Preface

This text deals particularly with nonlinear differential equations which are examined with the help of regular variation. It was created within the A-Math-Net project, and its primary purpose is to serve as a textbook for (almost) graduate students. But it can be useful also for experts or for anybody who is interested in asymptotic theory of differential equations.

The theory of regularly varying functions has been shown very useful in some fields of qualitative theory of differential equations, see, in particular, the important monograph by Marić [105] summarizing themes in the research up to the year 2000. This book served as an excellent source for our text, but of course, much material is taken also from other sources. Actually, our treatise includes many of the results that appeared after Marić's book. Moreover, it reveals new relations among various results, and revise some of them.

In view of the main aim of this text, we cannot give a complete treatment with all details. We rather focus on presenting a wide variety of methods which shows how powerful tool regular variation is. For reference purposes we present brief comprehensive surveys of sub-themes.

The first chapter summarizes useful information about regularly varying functions and related concepts. Our style of treatment with this topic is affected by requirements on applications in differential equations.

Although this text is focused on nonlinear equations, there is a big part (Chapter 2) which deals with linear differential equations. We offer a comprehensive survey of the results in which linear differential equations are studied in the framework of regular variation. Proofs are given only exceptionally, but we present many comments which sometimes include a description of the main ideas. The objective of this chapter is multiple. Some of the linear results are used in the nonlinear theory, thus we can easily refer them. Some of the results are extended to a nonlinear case, thus we can easily make a comparison; at the same time, some of the statements in Chapter 2 may serve as a motivation for an extension (which has not been made yet) to a nonlinear case. Moreover, our survey includes also the results which are not contained in the above mentioned Marić's book (especially the recent ones), and we point out relations among various results.

Chapter 3 deals with second-order half-linear differential equations in the framework of regular variation. The most of the results can be understood as a half-linear extension of some of the statements from Chapter 2. But this part offers more, especially as far as the methods are concerned. Indeed, many steps in the proofs require a quite new approach or at least a highly nontrivial modification comparing with the existing linear case.

Emden-Fowler type equations are examined in Chapter 4. We consider various types of such objects, second order equations, higher order equations, but also systems. Older as well as recent results are presented. Concerning recent results, because of their big amount, we give only a brief (but quite comprehensive) survey which is followed by a detailed description of several selected results. We try to make make a selection in such a way which shows a variety of typical approaches.

Chapter 5 is devoted to investigation of some other nonlinear differential equations where the theory of regular variation has been shown to be helpful. We deal with equations which involve a generalized Laplacian, partial differential equations, a class of third order nonlinear equations, perturbed first order equations, and second order nearly linear equations.

The last chapter briefly discusses utilization of regular variation in some other differential (or integral) equations; among others, equations with deviating arguments are mentioned. Further, it offers a short survey of the literature devoted to examination of difference equations, q-difference equations, and dynamic equations on time scales in the framework of regular variation.

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Pdf version of this text can be found on `http://www.amathnet.cz/` or on `users.math.cas.cz/~rehak/ndefrv`.

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Notation and convention

- $\mathcal{RV}(\vartheta)$: regularly varying functions (at infinity) of index ϑ [Section 1.1]
- $\mathcal{SV}$: slowly varying functions (at infinity) [Section 1.1]
- $\mathcal{RB}$: regularly bounded functions (at infinity) [Section 1.1]
- $\mathcal{RPV}(\pm\infty)$: rapidly varying functions (at infinity) of index $\pm\infty$ [Section 1.1]
- $\mathcal{NRV}(\vartheta)$: normalized regularly varying functions (at infinity) of index ϑ [Section 1.1]
- $\mathcal{NSV}(\vartheta)$: normalized slowly varying functions (at infinity) of index ϑ [Section 1.1]
- tr-$\mathcal{RV}(\vartheta)$: trivial regularly varying functions (at infinity) of index ϑ [Section 1.1]
- $\mathcal{SV}_0$: slowly varying functions at zero, similarly for $\mathcal{RV}_0(\vartheta)$, $\mathcal{RPV}_0(\pm)$, $\mathcal{RB}_0$, tr-$\mathcal{RV}_0(\vartheta)$ [Section 1.1]
- $\mathcal{RV}$: the class of all regularly varying functions (at infinity), similarly for $\mathcal{RV}_0$, $\mathcal{RPV}$, $\mathcal{RPV}_0$, tr-$\mathcal{RV}$ tr-$\mathcal{RV}_0$;
  "$\mathcal{RV}$" sometimes may mean the abbreviation of "regularly varying", similarly for $\mathcal{SV}$, $\mathcal{RPV}$, $\mathcal{RB}$ etc. [Section 1.1]
- Karamata functions: $\mathcal{RV} \cup \mathcal{RPV}$ [Section 1.1]
- $f^{\leftarrow}$: generalized inverse of f [Section 1.1]
- $\Pi(w)$: class Π in the de Haan sense with auxiliary function w [Section 1.2]
- $\Gamma(w)$: class Γ in the de Haan sense with auxiliary function w [Section 1.2]
- $\Pi\mathrm{R}_2(w, z)$: Omey-Willekens type functions [Section 1.2]
- $\mathcal{BSV}$: Beurling slowly varying functions [Section 1.2]
- $\mathcal{SN}$: self-neglecting functions [Section 1.2]
- $\mathcal{RV}_\omega(\vartheta)$: (generalized) regularly varying functions (at infinity) with respect to ω of index ϑ, similarly for $\mathcal{SV}_\omega$, $\mathcal{RPV}_\omega(\pm\infty)$, $\mathcal{RB}_\omega$, $\mathcal{NRV}_\omega$ [Subsection 1.3.1]
- L_f: slowly varying component of $f \in \mathcal{RV}$, i.e., $L_f(t) = f(t)/t^\vartheta$ for $f \in \mathcal{RV}(\vartheta)$.
- $\sim, \asymp, o, O$: For eventually positive functions f, g, we denote:

  $f(t) \sim g(t)$ if $\lim_{t\to\infty} f(t)/g(t) = 1$,
  $f(t) \asymp g(t)$ if $\exists c_1, c_2 \in (0, \infty)$ s. t. $c_1 g(t) \le f(t) \le c_2 g(t)$ for large t,
  $f(t) = o(g(t))$ if $\lim_{t\to\infty} f(t)/g(t) = 0$,

$$f(t) = O(g(t)) \text{ if } \exists c \in (0,\infty) \text{ s. t. } f(t) \le c g(t) \text{ for large } t.$$

- We adopt the usual conventions: $\prod_{j=k}^{k-1} u_j = 1$ and $\sum_{j=k}^{k-1} u_j = 0$.
- Φ, Φ_λ: The notation $\Phi(u)$ is typically used in connection with half-linear equations and means $\Phi(u) = |u|^{\alpha-1} \operatorname{sgn} u$ with $\alpha > 1$. The notation $\Phi_\lambda(u)$ is typically used in connection with Emden-Fowler type equations and means $\Phi_\lambda(u) = |u|^\lambda \operatorname{sgn} u$ with $\lambda > 0$.

  Notice that the powers are shifted by 1, but since α starts from 1 while λ starts from 0, the functions are practically the same. We decide to use such convention in accordance with some literature.

Chapter 1

Regular variation

The subject of regular variation as we use the term today was initiated by Jovan Karamata in a famous paper of 1930 [74], see also [73, 75], though preliminary or partial treatments may be found in earlier work of Landau in 1911, Valiron in 1913, Pólya in 1917, Schmidt in 1925, and others.

In its basic form, regular variation may be viewed as the study of relations such as

$$\frac{f(\lambda t)}{f(t)} \to g(\lambda) \in (0, \infty) \quad (t \to \infty) \quad \forall \lambda > 0,$$

together with its numerous ramifications. This study is referred to as Karamata theory. More general than the relation above is

$$\frac{f(\lambda t) - f(t)}{g(t)} \to h(\lambda) \in \mathbb{R} \quad (t \to \infty) \quad \forall \lambda > 0.$$

The study of relations of this kind is referred to as de Haan theory.

Mathematically, regular variation is essentially a field in classical real variable theory, together with its applications in integral transforms – Tauberian theorems, probability theory, analytic number theory, complex analysis, differential equations, and elsewhere.

Our style of dealing with regular variation in this chapter is designed for the purpose to study asymptotic behavior of differential equations.

A quite comprehensive treatment of regular variation can be found in the book by Bingham, Goldie, and Teugels [14]; much of material presented in this chapter can be found (and is proved) in that book. Other main sources for this chapter are the book by Seneta [156], the book by Geluk and de Haan [47], and the thesis by de Haan [50]. Several useful concepts and statements are taken from some papers on differential equations.

1.1 Karamata theory

1.1.1 Basic concepts

We start with two essential definitions.

Definition 1.1. A measurable function $f : [a, \infty) \to (0, \infty)$ is called *regularly varying (at infinity) of index* ϑ if

$$\lim_{t\to\infty} \frac{f(\lambda t)}{f(t)} = \lambda^{\vartheta} \tag{1.1}$$

for every $\lambda > 0$; we write $f \in \mathcal{RV}(\vartheta)$. The class of all regularly varying functions is denoted as

$$\mathcal{RV} = \bigcup_{\vartheta\in\mathbb{R}} \mathcal{RV}(\vartheta).$$

Definition 1.2. A measurable function $L : [a, \infty) \to (0, \infty)$ is called *slowly varying (at infinity)* if

$$\lim_{t\to\infty} \frac{L(\lambda t)}{L(t)} = 1; \tag{1.2}$$

we write $L \in \mathcal{SV}$.

A slowly varying function is customarily denoted by L because of the first letter of the French "lentement" which means "slowly"; note that the foundation papers by Karamata were written in French.

It is clear that $\mathcal{SV} = \mathcal{RV}(0)$. Consequently, the set of slowly varying functions forms a subset of the set of regularly varying ones. This however might be somehow misleading statement, since the class of slowly varying functions is the one which presents itself, due to wealth of interesting properties, as a major novelty in the classical analysis and applications. In the sequel the term "regularly varying functions" sometimes will include the slowly varying ones and sometimes not. The context, however, will prevent any ambiguity.

It is known that the conditions in the definition of $\mathcal{RV}$ functions can be weakened. Indeed, the limit relation in (1.1) is sufficient to hold only for λ in a set of positive measure and then the regular variation follows. Moreover, if the limit

$$\lim_{t\to\infty} \frac{f(\lambda t)}{f(t)} = g(\lambda) \in (0, \infty)$$

exists for λ in a set of positive measure, then g necessarily takes the form $g(\lambda) = \lambda^{\vartheta}$, where ϑ is some real number.

It is easy to show that $f \in \mathcal{RV}(\vartheta)$, $\vartheta \in \mathbb{R}$, if and only if it can be written in the form

$$f(t) = t^{\vartheta} L(t), \quad \text{where } L \in \mathcal{SV}. \tag{1.3}$$

Thus, for most purposes, to study regular variation, it suffices to study the properties of slowly varying functions.

Here are some examples of slowly varying functions:

$$
\begin{aligned}
L(t) &= \prod_{i=1}^{n} (\ln_i t)^{\mu_i}, \text{ where } \ln_i t = \ln \ln_{i-1} t \text{ and } \mu_i \in \mathbb{R},\\
L(t) &= \exp\left\{ \prod_{i=1}^{n} (\ln_i t)^{\nu_i} \right\}, \text{ where } 0 < \nu_i < 1,\\
L(t) &= 2 + \sin(\ln_2 t),\\
L(t) &= (\ln \Gamma(t))/t,\\
L(t) &= \frac{1}{t} \int_a^t \frac{1}{\ln s}\, ds,\\
L(t) &= \exp\left\{ (\ln t)^{\frac{1}{3}} \cos(\ln t)^{\frac{1}{3}} \right\}.
\end{aligned}
$$

The last example shows the $\mathcal{SV}$ function which exhibits "infinite oscillation", i.e., $\liminf_{t\to\infty} L(t) = 0$, $\limsup_{t\to\infty} L(t) = \infty$. This phenomenon can be somewhat in a contrast with the intuitive notion of a "slowly varying" behavior and it reveals that the class $\mathcal{RV}$ includes a wide variety of functions. In particular, slowly varying functions do not need to be monotone eventually. It is clear that the power function t^ϑ is a trivial example of regularly varying function (of index ϑ), but for $\vartheta \neq 0$, we have $t^\vartheta \notin \mathcal{SV}$. The exponential functions $\exp t, \exp(-t)$ are not regularly varying at all. But, for example, $1 + \exp(-t)$ is slowly varying. Undamped oscillatory functions such as $2 + \sin t$ are not slowly or regularly varying. It is interesting to observe that while $2 + \sin(\ln t)$ is not slowly varying, the function $2 + \sin(\ln_2 t)$ is slowly varying.

Some authors speak about the class of *trivial regularly varying function of index* ϑ, we write tr-$\mathcal{RV}(\vartheta)$, where $f \in$ tr-$\mathcal{RV}(\vartheta)$ if

$$f(t) \sim Ct^\vartheta \quad \text{as } t \to \infty,$$

C being some positive constant; we consider a positive measurable f. We denote

$$\text{tr-}\mathcal{RV} = \bigcup_{\vartheta \in \mathbb{R}} \text{tr-}\mathcal{RV}(\vartheta).$$

We have, $f \in$ tr-$\mathcal{RV}(\vartheta)$, if and only if

$$f(t) = t^\vartheta h(t) \ \text{ where } h(t) \sim C \in (0, \infty) \text{ as } t \to \infty,$$

cf. (1.3). It is clear that tr-$\mathcal{RV} \subset \mathcal{RV}$. Thus, regular variation of a function can be understood as a (one-sided, local) asymptotic property which arises out of trying to extend in a logical and useful manner the class of functions whose asymptotic behavior is that of a power function, to functions where asymptotic behavior is that of a power function multiplied by a factor which varies "more slowly" than a power function. As examples of slowly varying functions show, such an extension is far from being trivial.

We have defined regular variation at infinity. Of course, this is not the only possibility. A measurable function $f:(0,a]\to(0,\infty)$ is said to be *regularly varying at zero of index* ϑ if $\lim_{t\to 0+}\frac{f(\lambda t)}{f(t)}=\lambda^\vartheta$ for every $\lambda>0$; we write $f\in\mathcal{RV}_0(\vartheta)$. Since regular variation of $f(\cdot)$ at zero of index ϑ means in fact regular variation of $f(1/t)$ at infinity of index $-\vartheta$, properties of $\mathcal{RV}_0$ functions can be easily deduced from theory of $\mathcal{RV}$ functions. Regular variation can now be defined at any finite point by shifting the origin of the function to this point. It is thus apparent that it suffices to develop the theory of regular variation at infinity, which we shall do, frequently omitting the words "at infinity" in the sequel. For example, $-\ln t$ is slowly varying at $t=0+$, $\ln t$ is slowly varying at $t=1+$, $-\ln t$ is slowly varying at $t=1-$, and $-\ln(1-t)$ is slowly varying at $t=1$.

In connection with investigation of solutions to some differential equations, the concept of nearly regularly varying functions was introduced: If a positive continuous function f satisfies $f(t)\asymp g(t)$ as $t\to\infty$ for some $g\in\mathcal{RV}(\vartheta)$, then f is called a *nearly regularly varying function of index* ϑ.

1.1.2 Uniform convergence and representation

The following result (the so-called Uniform Convergence Theorem) is one of the most fundamental theorems in the theory. Many important properties of $\mathcal{RV}$ functions follow from it.

Theorem 1.1. *If $f\in\mathcal{RV}(\vartheta)$, then the relation* (1.1) *(and so* (1.2)*) holds uniformly on each compact λ-set in* $(0,\infty)$.

The second fundamental result is the following Representation Theorem. It follows from the previous result and vice versa.

Theorem 1.2. *A function L is slowly varying if and only if it has the form*

$$L(t)=\varphi(t)\exp\left\{\int_a^t\frac{\psi(s)}{s}\,ds\right\},\tag{1.4}$$

$t\ge a$, for some $a>0$, where φ,ψ are measurable with $\lim_{t\to\infty}\varphi(t)=C\in(0,\infty)$ *and* $\lim_{t\to\infty}\psi(t)=0$.

Since L,φ,ψ may be altered at will on finite intervals, the value of a is unimportant; if $a=0$ one can take $\psi\equiv 0$ on a neighborhood of 0 to avoid divergence of the integral at the origin. In view of (1.3), a function $f\in\mathcal{RV}(\vartheta)$ may be written as

$$f(t)=t^\vartheta\varphi(t)\exp\left\{\int_a^t\frac{\psi(s)}{s}\,ds\right\},\tag{1.5}$$

where φ and ψ are as in the theorem. Alternatively, $f\in\mathcal{RV}(\vartheta)$ if and only if it has the representation

$$f(t)=\varphi(t)\exp\left\{\int_a^t\frac{\delta(s)}{s}\,ds\right\},\tag{1.6}$$

where δ is measurable with $\lim_{t\to\infty}\delta(t) = \vartheta$.

The Karamata representation (1.4) is essentially non-unique: within limits, one may always adjust one of $\varphi(\cdot)$, $\psi(\cdot)$, making a compensating adjustment to the other. It turns out that the function ψ may be arbitrarily smooth, but the smoothness properties attainable for φ ale limited by those present in L. However, replacing $\varphi(t)$ by its limit $C \in (0,\infty)$, we obtain a $\mathcal{SV}$ function which is asymptotic to the original one, but with much enhanced properties; this topic will be discussed also later.

As already indicated above, from some points of view (for instance, the measuring of scales of growth like in studying asymptotic behavior of relevant functions), slowly varying functions are of interest only to within an asymptotic equivalence. We then lose nothing by restricting attention to the case $\varphi(t) \equiv C$ in (1.4) or (1.5) or (1.6). The following definition is pertinent to this situation.

Definition 1.3. The regularly varying function of index ϑ

$$f(t) = t^{\vartheta} C \exp\left\{\int_a^t \frac{\psi(s)}{s}\,\mathrm{d}s\right\}, \tag{1.7}$$

$\lim_{t\to\infty}\psi(t) = 0$, $C \in (0,\infty)$, is called *normalized*; we write $f \in \mathcal{NRV}(\vartheta)$. The set of normalized slowly varying functions, i.e., $\mathcal{NRV}(0)$, is denoted as $\mathcal{NSV}$.

For $L \in \mathcal{NSV}$, $\psi(t) = tL'(t)/L(t)$ almost everywhere. Conversely, given a function L with $\psi(t) := tL'(t)/L(t)$ continuous and $o(1)$ at infinity, we may integrate to obtain (1.7), showing L to be normalized slowly varying.

The class $\mathcal{NSV}$ coincides with the so-called Zygmund class which is defined as follows: A positive measurable function f belongs to the *Zygmund class* if, for every $\vartheta > 0$,

$t^{\vartheta} f(t)$ is ultimately increasing and $t^{-\vartheta} f(t)$ is ultimately decreasing.

1.1.3 Karamata theorem

As we will see later, the results of this subsection are extremely useful in applications to the theory of differential equations.

Theorem 1.3 (Karamata's theorem; direct half)**.** *If $L \in \mathcal{SV}$, then*

$$\int_t^{\infty} s^{\zeta} L(s)\,\mathrm{d}s \sim \frac{1}{-\zeta-1} t^{\zeta+1} L(t) \tag{1.8}$$

provided $\zeta < -1$, and

$$\int_a^t s^{\zeta} L(s)\,\mathrm{d}s \sim \frac{1}{\zeta+1} t^{\zeta+1} L(t) \tag{1.9}$$

provided $\zeta > -1$. The integral $\int_a^{\infty} L(s)/s\,\mathrm{d}s$ may or may not converge. The function

$$\tilde{L}(t) = \int_t^{\infty} \frac{L(s)}{s}\,\mathrm{d}s \quad \textit{resp.} \quad \tilde{L}(t) = \int_a^t \frac{L(s)}{s}\,\mathrm{d}s \tag{1.10}$$

is a new $\mathcal{SV}$ function and $L(t)/\tilde{L}(t) \to 0$ as $t \to \infty$.

We shall see in Section 1.2 that actually the integral in (1.10) is, more precisely, a de Haan function, to be defined in that section. For later use, it can be convenient to write the above theorem in a slightly different form.

Theorem 1.4. *Let $f \in \mathcal{RV}(\zeta)$ and be locally bounded in $[a, \infty)$. Then*

(i) For any $\sigma < -(\zeta + 1)$ (and for $\sigma = -(\zeta + 1)$ if $\int_a^\infty s^{-(\zeta+1)} f(s)\,\mathrm{d}s < \infty$),

$$t^{\sigma+1} f(t) \Big/ \int_t^\infty s^\sigma f(s)\,\mathrm{d}s \to -(\sigma + \zeta + 1) \text{ as } t \to \infty;$$

(ii) for any $\sigma \geq -(\zeta + 1)$,

$$t^{\sigma+1} f(t) \Big/ \int_a^t s^\sigma f(s)\,\mathrm{d}s \to \sigma + \zeta + 1 \text{ as } t \to \infty.$$

The above theorems tell us in detail how $\mathcal{SV}$ functions behave when multiplied by powers and integrated. It is a remarkable fact that such behavior can only arise in the case of regular variation.

Theorem 1.5 (Karamata's theorem; converse half)**.** *Let f be positive and locally integrable in $[a, \infty)$.*

(i) If for some $\sigma < -(\zeta + 1)$,

$$t^{\sigma+1} f(t) \Big/ \int_t^\infty s^\sigma f(s)\,\mathrm{d}s \to -(\sigma + \zeta + 1) \text{ as } t \to \infty,$$

then $f \in \mathcal{RV}(\zeta)$. (ii) If for some $\sigma > -(\zeta + 1)$,

$$t^{\sigma+1} f(t) \Big/ \int_a^t s^\sigma f(s)\,\mathrm{d}s \to \sigma + \zeta + 1 \text{ as } t \to \infty,$$

then $f \in \mathcal{RV}(\zeta)$.

1.1.4 Monotonicity

Various aspects of the theory of regular variation are simplified if the functions in questions are assumed monotone. For instance:

- If L is eventually positive and monotone and there exists $\lambda_0 \in (0, \infty) \setminus \{1\}$ with

$$\lim_{t \to \infty} \frac{L(\lambda_0 t)}{L(t)} = 1,$$

  then $L \in \mathcal{SV}$.

- If f is eventually positive and monotone and

$$\lim_{t\to\infty} \frac{f(\lambda t)}{f(t)} = g(\lambda) \in (0, \infty)$$

for all λ in some dense subset of $(0, \infty)$ or just for $\lambda = \lambda_1, \lambda_2$ with $(\ln \lambda_1)/\ln \lambda_2$ finite and irrational, then $f \in \mathcal{RV}$.

- A monotone positive function f is $\mathcal{RV}$ if and only if there exist two sequences $\{c_n\}$ and $\{a_n\}$ of positive numbers with

$$\lim_{n\to\infty} \frac{c_{n+1}}{c_n} = 1, \qquad \lim_{n\to\infty} a_n = \infty$$

such that for all positive λ

$$\lim_{n\to\infty} c_n f(\lambda a_n)$$

exists, is positive and finite.

1.1.5 Further selected properties of $\mathcal{RV}$ functions

- If $f \in \mathcal{RV}(\vartheta)$, then $\ln f(t)/\ln t \to \vartheta$ as $t \to \infty$. It then clearly implies that $\lim_{t\to\infty} f(t) = 0$ provided $\vartheta < 0$, and $\lim_{t\to\infty} f(t) = \infty$ provided $\vartheta > 0$.

- If $f \in \mathcal{RV}(\vartheta)$, then $f^\alpha \in \mathcal{RV}(\alpha\vartheta)$ for every $\alpha \in \mathbb{R}$.

- If $f_i \in \mathcal{RV}(\vartheta_i)$, $i = 1, 2$, $f_2(t) \to \infty$ as $t \to \infty$, then $f_1 \circ f_2 \in \mathcal{RV}(\vartheta_1\vartheta_2)$.

- If $f_i \in \mathcal{RV}(\vartheta_i)$, $i = 1, 2$, then $f_1 + f_2 \in \mathcal{RV}(\max\{\vartheta_1, \vartheta_2\})$.

- If $f_i \in \mathcal{RV}(\vartheta_i)$, $i = 1, 2$, then $f_1 f_2 \in \mathcal{RV}(\vartheta_1 + \vartheta_2)$.

- If $f_1, \ldots, f_n \in \mathcal{RV}$, $n \in \mathbb{N}$, and $R(x_1, \ldots, x_n)$ is a rational function with positive coefficients, then $R(f_1, \ldots, f_n) \in \mathcal{RV}$.

- If $L \in \mathcal{SV}$ and $\vartheta > 0$, then $t^\vartheta L(t) \to \infty$, $t^{-\vartheta} L(t) \to 0$ as $t \to \infty$.

- Let f be eventually positive and differentiable, and let

$$\lim_{t\to\infty} \frac{t f'(t)}{f(t)} = \vartheta.$$

Then $f \in \mathcal{NRV}(\vartheta)$.

- If $f \in \mathcal{RV}(\vartheta)$ with $\vartheta \le 0$ and $f(t) = \int_t^\infty g(s)\,\mathrm{d}s$ with g nonincreasing, then

$$\frac{-t f'(t)}{f(t)} = \frac{t g(t)}{f(t)} \to -\vartheta \ \text{ as } t \to \infty.$$

- If $f \in \mathcal{RV}(\vartheta)$ with $\vartheta \geq 0$ and $f(t) = f(t_0) + \int_{t_0}^{t} g(s)\,\mathrm{d}s$ with g monotone, then

$$\frac{tf'(t)}{f(t)} = \frac{tg(t)}{f(t)} \to \vartheta \quad \text{as } t \to \infty.$$

- (Similar to the above ones but much earlier results.) If the derivative of $L \in \mathcal{SV}$ is monotone, then $tL'(t)/L(t) \to 0$ as $t \to \infty$.

  Suppose that $F \in \mathcal{RV}(\vartheta)$ with $\vartheta \in \mathbb{R}$ and that there exists a monotone function f such that for all positive t, $F(t) = \int_0^t f(s)\,\mathrm{d}s$. Then

$$\lim_{t\to\infty} \frac{tf(t)}{F(t)} = \vartheta.$$

  Hence, for $\vartheta \neq 0$, it holds $f \operatorname{sgn} \vartheta \in \mathcal{RV}(\vartheta - 1)$.

- *(Almost monotonicity)* For a positive measurable function L it holds: $L \in \mathcal{SV}$ if and only if, for every $\vartheta > 0$, there exist a (regularly varying) nondecreasing function F and a (regularly varying) nonincreasing function G with

$$t^{\vartheta}L(t) \sim F(t) \quad t^{-\vartheta}L(t) \sim G(t) \qquad \text{as } t \to \infty.$$

  In particular, if $L \in \mathcal{SV}$ and $\vartheta > 0$, $t^{\vartheta}L(t)$ is asymptotic to a nondecreasing function, $t^{-\vartheta}L(t)$ to a nonincreasing one. Or, a regularly varying function of index $\vartheta \neq 0$ is almost monotone. Recall that a positive function f is called almost increasing on $[a, \infty)$ if for some constant $M > 0$, $f(s) \geq f(Mt)$, $s \geq t \geq a$. We may write $f(t) = O(\inf_{s\geq t} f(t))$ or even $f(t) \asymp \inf_{s\geq t} f(s)$. Similarly we define an almost decreasing function f; then we have $f(t) \asymp \sup_{s\geq t} f(t)$.

- *(Asymptotic inversion)* If $g \in \mathcal{RV}(\vartheta)$ with $\vartheta > 0$, then there exists $g \in \mathcal{RV}(1/\vartheta)$ with

$$f(g(t)) \sim g(f(t)) \sim t \quad \text{as } t \to \infty.$$

  Here g (an "asymptotic inverse" of f) is determined uniquely up to asymptotic equivalence. One version of g is the *generalized inverse*

$$f^{\leftarrow}(t) := \inf\{s \in [a, \infty) : f(s) > t\}.$$

- *(de Bruijn conjugacy)* If $L \in \mathcal{SV}$, there exists $L^{\#} \in \mathcal{SV}$, unique up to asymptotic equivalence, with

$$L(t)L^{\#}(tL(t)) \to 1, \ \ L^{\#}(t)L(tL^{\#}(t)) \to 1$$

  as $t \to \infty$. Then $L^{\#\#} \sim L$. The function $L^{\#}$ is the *Bruijn conjugate* of L; $(L, L^{\#})$ is a *conjugate pair*.

- *(Smooth variation)* First we give definition of the class $\mathcal{SRV}$: A positive function *varies smoothly with index* $\vartheta \in \mathbb{R}$, we write $f \in \mathcal{SRV}(\vartheta)$, if $h(t) := \ln f(e^t)$ is C^∞, and
$$h'(t) \to \vartheta, \;\; h^{(n)}(t) \to 0 \;\; (n = 2, 3, \dots) \;\text{ as } t \to \infty. \tag{1.11}$$

  If $f \in \mathcal{RV}(\vartheta)$, then there exist $f_1, f_2 \in \mathcal{SRV}(\vartheta)$ with $f_1 \sim f_2$ and $f_1 \le f \le f_2$ on some neighborhood of infinity. In particular, if $f \in \mathcal{RV}(\vartheta)$, there exists $g \in \mathcal{SRV}(\vartheta)$ with $g \sim f$. Thus for many purposes it suffices to restrict attention to the smoothly varying case.

  Note that if $f \in \mathcal{SRV}(\vartheta)$, then $tf'(t)/f(t) = h'(\ln t) \to \vartheta$ and is continuous, whence $f \in \mathcal{NRV}(\vartheta)$. Condition (1.11) is equivalent to
$$\frac{t^n f^{(n)}(t)}{f(t)} \to \vartheta(\vartheta - 1) \cdots (\vartheta - n + 1),$$
  $n = 1, 2, \dots,$ as $t \to \infty$. Smooth variation is well adapted to the processes of integration and differentiation. If $f \in \mathcal{SRV}(\vartheta)$, $\vartheta \neq 0$, then $|f'| \in \mathcal{SRV}(\vartheta - 1)$. If $f \in \mathcal{SRV}(\vartheta)$, then for $\vartheta > -1$, $\int_a^t f(s)\,\mathrm{d}s \in \mathcal{SRV}(\vartheta + 1)$, and for $\vartheta < -1$, $\int_t^\infty f(s)\,\mathrm{d}s \in \mathcal{SRV}(\vartheta + 1)$.

  Note that if $L \in \mathcal{SV}$, then there exists another, infinitely differentiable, $\mathcal{SV}$ function L_1 such that $L_1(t) \sim L(t)$ as $t \to \infty$ and $L_1(n) = L(n)$ for all large $n \in \mathbb{N}$.

  If $f \in \mathcal{SRV}(\vartheta)$ and $\vartheta \notin \{0, 1, 2, \dots\}$, each derivative will ultimately have constant sign, so $|f^{(n)}| \in \mathcal{SRV}(\vartheta - n)$. In particular, for $\vartheta > 1$ the first $[\vartheta] - 1$ derivatives will be ultimately convex and the $[\vartheta]$-th derivative ultimately concave. A related result is the following: If $f \in \mathcal{RV}(\vartheta)$, $\vartheta \notin \{0, 1, 2, \dots\}$, then there exists a C^∞-function g, all of whose derivatives are monotone, with $f(t) \sim g(t)$.

- If $f \in \mathcal{RV}(\vartheta)$, $\vartheta \in \mathbb{R}$, then for all sequences $\{a_n\}, \{b_n\}$ of positive numbers with $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \infty$ and $\lim_{n\to\infty} a_n/b_n = \lambda \in (0, \infty)$, we have
$$\lim_{n\to\infty} \frac{f(a_n)}{f(b_n)} = \lambda^\vartheta. \tag{1.12}$$

  If $\vartheta \in \mathbb{R} \setminus \{0\}$, the conclusion is also true for $\lambda = 0$ and $\lambda = \infty$.

- (See [2].) Let $L \in \mathcal{SV}$ and assume that $\int_a^\infty t^\eta |f(t)|\,\mathrm{d}t$ converge for some $\eta > 0$. Then $\int_a^\infty f(t)L(xt)\,\mathrm{d}t$ exists and
$$\int_a^\infty f(t)L(xt)\,\mathrm{d}t \sim L(x) \int_a^\infty f(t)\,\mathrm{d}t$$
  as $x \to \infty$.

1.1.6 Regularly bounded functions

The main limitation of the theory so far developed is the need to assume the existence of the limit $\lim_{t\to\infty}\frac{f(\lambda t)}{f(t)}$. A natural and useful generalization of regular variation is the concept of regularly bounded functions (or O-regularly varying functions) introduced by Avakumović in 1935.

Definition 1.4. A measurable function $f : [a,\infty) \to (0,\infty)$ is called *regularly bounded* if

$$0 < \liminf_{t\to\infty}\frac{f(\lambda t)}{f(t)} \le \limsup_{t\to\infty}\frac{f(\lambda t)}{f(t)} < \infty \quad \text{for every } \lambda \ge 1; \tag{1.13}$$

we write $f \in \mathcal{RB}$.

Equivalently, $f \in \mathcal{RB}$ can be defined via $\limsup_{t\to\infty}\frac{f(\lambda t)}{f(t)} < \infty$ for all $\lambda > 0$. The set of regularly bounded functions at zero (i.e., we take the limit as $t \to 0$) is denoted as $\mathcal{RB}_0$.

Clearly any $\mathcal{RV}$ function is $\mathcal{RB}$. Any positive and measurable function which is bounded away from both 0 and ∞ satisfies this definition; thus various simple oscillating functions noted hitherto as not being regularly varying, such as $2+\sin t$ and $t^\gamma(1+\alpha\sin(\ln t))$ with α small, are regularly bounded, though e^t still not. It is evident that if measurability is strengthened to monotonicity one of the bounds in $m \le f(\lambda t)/f(t) \le M$ (such inequalities can alternatively define $\mathcal{RB}$) is automatically satisfied. Further, for instance, if f is nondecreasing, instead of $\limsup_{t\to\infty}\frac{f(\lambda t)}{f(t)} < \infty$ for all $\lambda > 1$, it then is sufficient to require $\limsup_{t\to\infty}\frac{f(\lambda_0 t)}{f(t)} < \infty$ for some $\lambda_0 > 1$.

Here are selected properties of $\mathcal{RB}$ functions:

- *(Uniform convergence theorem for $\mathcal{RB}$)* If $f \in \mathcal{RB}$, then, for every $\Lambda > 1$, (1.13) holds uniformly in $\lambda \in [1,\Lambda]$.

- *(Representation theorem for $\mathcal{RB}$)* A function f is regularly bounded if and only if it has the representation

  $$f(t) = \exp\left\{\xi(t) + \int_a^t \frac{\eta(s)}{s}\,\mathrm{d}s\right\},$$

  $t \ge a$, where ξ and η are bounded and measurable on $[a,\infty)$. If $\xi(t) \equiv \text{const}$ in the representation, then f is referred to as a *normalized regularly bounded* function.

- A positive continuous function f is regularly bounded if and only if there exist $\gamma,\delta \in \mathbb{R}$, $\gamma > \delta$, such that $t^\gamma f(t)$ is eventually almost increasing and $t^\delta f(t)$ is eventually almost decreasing.

- It holds, $f \in \mathcal{RB}$ if and only if there exists $\delta \in \mathbb{R}$ such that

  $$\int_a^t s^{\delta-1} f(s)\,\mathrm{d}s \asymp t^\delta f(t) \quad \text{as } t \to \infty.$$

∘ It holds, $f \in \mathcal{RB}$ if and only if there exists $\gamma \in \mathbb{R}$ such that

$$\int_t^\infty s^{\gamma-1} f(s)\,\mathrm{d}s \asymp t^\gamma f(t) \quad \text{as } t \to \infty.$$

1.1.7 Rapid variation

We proceed with another problem which naturally arises out: We examine functions for which the limit in (1.1) attains the extreme values.

Definition 1.5. A measurable function $f : [a, \infty) \to (0, \infty)$ is called *rapidly varying of index* ∞, we write $f \in \mathcal{RPV}(\infty)$, if

$$\lim_{t\to\infty} \frac{f(\lambda t)}{f(t)} = \begin{cases} 0 & \text{for } 0 < \lambda < 1, \\ \infty & \text{for } \lambda > 1, \end{cases} \tag{1.14}$$

and is called *rapidly varying of index* $-\infty$, we write $f \in \mathcal{RPV}(-\infty)$, if

$$\lim_{t\to\infty} \frac{f(\lambda t)}{f(t)} = \begin{cases} \infty & \text{for } 0 < \lambda < 1, \\ 0 & \text{for } \lambda > 1. \end{cases} \tag{1.15}$$

The class of all rapidly varying solutions is denoted as $\mathcal{RPV}$.

While $\mathcal{RV}$ functions behaved like power functions (up to a factor which varies "more slowly"), $\mathcal{RPV}$ functions have a behavior close to that of exponential functions. In particular, $e^t \in \mathcal{RPV}(\infty)$ and $e^{-t} \in \mathcal{RPV}(-\infty)$.

If, for $\lambda > 0$, we adopt the convention

$$\lambda^\infty = \begin{cases} 0 & \text{for } \lambda < 1 \\ 1 & \text{for } \lambda = 1 \\ \infty & \text{for } \lambda > 1 \end{cases} \qquad \lambda^{-\infty} = \begin{cases} \infty & \text{for } \lambda < 1 \\ 1 & \text{for } \lambda = 1 \\ 0 & \text{for } \lambda > 1 \end{cases}$$

then regular and rapid variation of f can be expressed in a unique formula $\lim_{t\to\infty} f(\lambda t)/f(t) = \lambda^\vartheta$, where $\vartheta \in \mathbb{R} \cup \{\pm\infty\}$.

We now present selected properties of $\mathcal{RPV}$ functions. Notice that some of the results for regularly varying functions have partial analogues for rapid variation.

∘ *(Uniform convergence theorem for $\mathcal{RPV}$)* If $f \in \mathcal{RPV}(\infty)$, then (1.14) holds uniformly in λ over all intervals $(0, M^{-1})$ and (M, ∞) for every $M > 1$.

∘ To establish (1.14), only $f(\lambda t)/f(t) \to \infty$ for $\lambda > 1$ has to be proved. Similarly for (1.15).

∘ Let f be positive and differentiable, and let there exist

$$\lim_{t\to\infty} \frac{t f'(t)}{f(t)} = \pm\infty.$$

Then $f \in \mathcal{RPV}$.

- If $f \in \mathcal{RPV}$ with f' increasing and $\lim_{t\to\infty} f(t) = 0$, then

$$\lim_{t\to\infty} \frac{-tf'(t)}{f(t)} = \infty.$$

- There is a connection between a slowly and rapidly varying function. Let f be positive, locally bounded, and (globally) unbounded on $[0,\infty)$. If $f \in \mathcal{SV}$, then $f^{\leftarrow} \in \mathcal{RPV}(\infty)$ (in fact, $\mathcal{RPV}(\infty)$ can be replaced here by a certain subclass, see [14, Theorem 2.4.7]). If $f \in \mathcal{RPV}(\infty)$, then $f^{\leftarrow} \in \mathcal{SV}$.

- If $\vartheta = \pm\infty$, the conclusion (1.12) is true for monotone function f and $c \in ([0,\infty) \cup \{\infty\}) \setminus \{1\}$.

- Suppose the function $f : (0,\infty) \to (0,\infty)$ is nonincreasing. If $f \in \mathcal{RPV}(-\infty)$, then for all $\vartheta \in \mathbb{R}$, $\int_1^\infty t^\vartheta f(t)\,\mathrm{d}t < \infty$ and

$$\lim_{t\to\infty} \frac{t^{\vartheta+1} f(t)}{\int_t^\infty s^\vartheta f(s)\,\mathrm{d}s} = \infty. \tag{1.16}$$

  If for some $\vartheta \in \mathbb{R}$ the integral $\int_a^\infty t^\vartheta f(t)\,\mathrm{d}t$ converges and (1.16) holds, then $f \in \mathcal{RPV}(-\infty)$.

  Suppose the function $f : (0,\infty) \to (0,\infty)$ is nondecreasing. If $f \in \mathcal{RPV}(\infty)$, then for all $\vartheta \in \mathbb{R}$ for which the integral $\int_0^1 t^\vartheta f(t)\,\mathrm{d}t$ converges, we have

$$\lim_{t\to\infty} \frac{t^{\vartheta+1} f(t)}{\int_0^t s^\vartheta f(s)\,\mathrm{d}s} = \infty. \tag{1.17}$$

  If for some $\vartheta \in \mathbb{R}$ the integral $\int_0^1 t^\vartheta f(t)\,\mathrm{d}t$ converges and (1.17) holds, then $f \in \mathcal{RPV}(\infty)$.

1.2 De Haan theory

In this section we work with somehow more general relations than the one defining regular variation, namely such as $(h(\lambda t) - h(t))/w(t) \to k(\lambda)$ as $t \to \infty$. Note that functions satisfying this relation (which leads to the class Π) were introduced by Bojanić and Karamata in 1963. They were rediscovered, and definitely studied by de Haan in his thesis of 1970 [50], who introduced and studied related class Γ which is also discussed below.

1.2.1 Class Π

The Karamata theory considered so far concerns asymptotic relations such as $f(\lambda t)/f(t) \to g(\lambda)$ as $t \to \infty$. Writing $h = \ln f$ and $k = \ln g$, this becomes

$$h(\lambda t) - h(t) \to k(\lambda) \quad \text{as } t \to \infty.$$

One can work instead with the more general relation

$$\frac{h(\lambda t)-h(t)}{w(t)} \to k(\lambda) \quad \text{as } t\to\infty \text{, for all } \lambda>0, \tag{1.18}$$

where $w:(0,\infty)\to(0,\infty)$ is called *auxiliary function* of f. By that a new class of functions — *de Haan class* — is introduced, see de Haan [50] and Geluk, de Haan [47]. The case $w\equiv 1$ leads to the Karamata case.

First note one context in which such relations naturally arise, namely the "limiting case" in the Karamata theorem. Here the converse half yields no assertion, while the direct half tells us that if L varies slowly and $h(t) := \int_a^t \frac{L(s)}{s}\,\mathrm{d}s$, then $\frac{h(t)}{L(t)} \to \infty$. Much more precise links between L and h exist, however, involving differencing, as the Uniform Convergence Theorem yields

$$\frac{h(\lambda t)-h(t)}{L(t)} = \int_1^\lambda \frac{L(tu)}{L(t)u}\,\mathrm{d}u \to \int_1^\lambda \frac{\mathrm{d}u}{u} = \ln\lambda$$

as $t\to\infty$.

The denominator w in (1.18) needs in general to be taken regularly varying. If $w\in\mathcal{RV}(\vartheta)$, then $k(\lambda)$, if it exists finite for all $\lambda>0$, has to be of the form

$$k(\lambda) = \begin{cases} \ln\lambda & \text{for } \vartheta=0,\\ \frac{\lambda^\vartheta-1}{\vartheta} & \text{for } \vartheta\neq 0.\end{cases}$$

If $\vartheta\neq 0$, nothing new is obtained (essentially we get regular variation). But $\vartheta=0$ leads to a new useful class. This class is, after taking absolute values, a proper subclass of the Karamata class $\mathcal{SV}$.

Definition 1.6. A measurable function $f\in[a,\infty)\to\mathbb{R}$ is said to belong to the class Π if there exists a function $w:(0,\infty)\to(0,\infty)$ such that for $\lambda>0$

$$\lim_{t\to\infty}\frac{f(\lambda t)-f(t)}{w(t)} = \ln\lambda; \tag{1.19}$$

we write $f\in\Pi$ or $f\in\Pi(w)$. The function w is called an *auxiliary function* for f.

Note that de Haan [50] studied the class $\Pi_C(w)$ of $\mathcal{SV}$ functions for which a positive and measurable function w exists such that for all $\lambda>0$,

$$\lim_{t\to\infty}\frac{f(\lambda t)-f(t)}{w(t)} = C\ln\lambda. \tag{1.20}$$

If $C\neq 0$, then w must be $\mathcal{SV}$. If $C=0$ the slow variation of w is assumed. Bingham et al. [14] studied classes of functions satisfying general asymptotic relations related to (1.20). For instance, for $w\in\mathcal{RV}(\vartheta)$, they consider the class of measurable f satisfying for all $\lambda\geq 1$

$$\lim_{t\to\infty}\frac{f(\lambda t)-f(t)}{w(t)} = Ck_\vartheta(\lambda), \quad \text{where } k_\vartheta(\lambda) = \int_1^\lambda u^{\vartheta-1}\mathrm{d}u.$$

The resulting theory (de Haan theory) is both a direct generalization of the Karamata theory above and what is needed to fill certain gaps, or boundary cases, in Karamata's main theorem. The term "second-order theory" is sometimes used for this study. The original motivation was probabilistic. De Haan class presents itself as very fruitful in various applications; as we will see later they include applications in differential equations.

Let us give several examples. The functions f defined by

$$\begin{aligned}
&f(t) = \ln t + o(1),\\
&f(t) = (\ln t)^{\alpha}(\ln_2 t)^{\beta} + o(\ln t)^{\alpha-1},\ \alpha > 0, \beta \in \mathbb{R},\\
&f(t) = \exp\{(\ln t)^{\gamma}\} + o(\ln t)^{\gamma-1}\exp\{(\ln t)^{\gamma}\},\ 0 < \gamma < 1,\\
&f(t) = t^{-1}\ln\Gamma(t) + o(1)
\end{aligned}$$

are in Π. The function

$$f(t) = 2\ln t + \sin\ln t$$

is in $\mathcal{SV}$, but not in Π.

Now we present selected properties of functions in the class Π.

- If $f \in \Pi$, then for $0 < c < d < \infty$ relation (1.19) holds uniformly for $\lambda \in [c, d]$.
- Auxiliary function is unique up to asymptotic equivalence.
- The statements $f \in \Pi$ and

  $$\lim_{t\to\infty} \frac{f(\lambda t) - f(t)}{f(t) - \frac{1}{t}\int_a^t f(s)\,\mathrm{d}s} = \ln\lambda$$

  for $\lambda > 0$ are equivalent.
- The statements $f \in \Pi$ and there exists $L \in \mathcal{SV}$ such that

  $$f(t) = L(t) + \int_a^t \frac{L(s)}{s}\,\mathrm{d}s \tag{1.21}$$

  are equivalent.
- If f satisfies (1.21), then $f \in \Pi(L)$. Hence

  $$L(t) \sim f(t) - \frac{1}{t}\int_a^t f(s)\,\mathrm{d}s \tag{1.22}$$

  as $t \to \infty$. If $f \in \Pi(L)$ is integrable on finite intervals of $(0, \infty)$, then (1.22) holds.
- If $f \in \Pi$, then $\lim_{t\to\infty} f(t) =: f(\infty) \le \infty$ exists. If the limit is infinite, then $f \in \mathcal{SV}$. If the limit is finite, then $f(\infty) - f(t) \in \mathcal{SV}$.

- If $-f_i \in \Pi(w_i)$, where f_i is eventually positive, $i = 1, 2$, then $-f_1 f_2 \in \Pi(f_1 w_2 + f_2 w_1)$.
- If $f \in \Pi(w)$, g is measurable and $\{f(t) - g(t)\}/w(t) \to c \in \mathbb{R}$ as $t \to \infty$, then $g \in \Pi(w)$.
- If $f \in \Pi(w)$, then for any $\varepsilon > 0$ there exist $s_0, M \in (0, \infty)$ such that for $s \geq s_0$, $t \geq 1$,
$$\left|\frac{f(st) - f(s)}{w(s)}\right| \leq Mt^{\varepsilon}.$$

For further properties see [14, 47, 50]. There are another classes, closely related to Π, which can also be important for our purposes. Geluk in [44] introduces Π-regular variation:

$$f \in \Pi\mathcal{RV}(\vartheta) \text{ if an only if } \frac{f(t)}{t^{\vartheta}} \in \Pi.$$

Further, in Omey, Willekens [134], the following class of functions was introduced. As we will see, it opens further possibilities in obtaining more precise information about certain solutions of certain differential equations.

Definition 1.7. Let $f : (0, \infty) \to \mathbb{R}$ be measurable. If there exist measurable functions w, z such that $z \in \mathcal{SV}$ and

$$\frac{f(\lambda t) - f(t) - w(t) \ln \lambda}{z(t)} \to H(\lambda) \quad \text{as } t \to \infty \text{ for } \lambda > 0,$$

for some function $H(\lambda)$, then we write $f \in \Pi\mathrm{R}_2(w, z)$.

Next we present selected information about the class $\Pi\mathrm{R}_2(w, z)$.

- If $z(t) = o(w(t))$, then $f \in \Pi\mathrm{R}_2(w, z)$ implies that $f \in \Pi$.
- *(Auxiliary concepts)* Consider functions h satisfying
$$h(\lambda t) - h(t) \sim k(\lambda) g(t)$$
as $t \to \infty$. If $g \in \mathcal{SV}$, the limit function $k(\lambda)$ can be characterized as follows: for each $\mu, \lambda > 0$, we have
$$k(\lambda\mu) = k(\lambda) + k(\mu).$$
Hence $k(\lambda) = c \ln \lambda$ for some real c. The corresponding class of functions h will be denoted by $\Pi\mathrm{V}(c, g)$. Note that $h \in \mathcal{RV}(\vartheta)$ if and only if $\ln h \in \Pi\mathrm{V}(\vartheta, 1)$. If $c \neq 1$, the class $\Pi\mathrm{V}(c, g)$ is the class Π.
- *(Representation theorem)* Suppose that $z \in \mathcal{SV}$ and f is locally bounded and define
$$G(t) = f(t) - \frac{1}{t} \int_0^t f(s)\, \mathrm{d}s,$$

$t > 0$. Then $f \in \Pi\mathrm{R}_2(w, z)$ if and only if there exists $c \in \mathbb{R}$ such that $G \in \Pi\mathrm{V}(c, z)$. Moreover, if $f \in \Pi\mathrm{R}_2(w, z)$, then

$$\lim_{t\to\infty} \frac{w(t) - G(t)}{z(t)} = c_0$$

exists and $f \in \Pi\mathrm{R}_2(G, z)$ with limit function $H(\lambda) = c \ln \lambda + \frac{1}{2}c(\ln \lambda)^2$.

1.2.2 Class Γ

For $\mathcal{RV}$ functions (generalized) inversion gives again an $\mathcal{RV}$ function. For nondecreasing unbounded functions in the class Π (which forms a proper subset of $\mathcal{SV}$) we obtain the following class by inversion; it is a useful subclass of rapidly varying functions.

Another view of understanding is an extension of the notion of $\mathcal{RV}$ functions defined by $\lim_{t\to\infty} f(\lambda t)/f(t) = \lambda^\vartheta$ in the sense that we consider the class of functions satisfying the following property: There exists a function $g : (0, \infty) \to (0, \infty)$ and $\vartheta \in \mathbb{R}$ such that

$$\lim_{t\to\infty} \frac{f\left(t\lambda^{g(t)}\right)}{f(t)} = \lambda^\vartheta$$

for all positive λ. First we confine our considerations to nondecreasing functions f and ask for a characterization of the class of functions for which this relation holds with $\vartheta > 0$. Without loss of generality we may take $\vartheta = 1$ (this only involves a trivial change in g).

It turns out to be more convenient to start with the following definition which is a mere transformation of the one just given.

Definition 1.8. A nondecreasing function $f : \mathbb{R} \to (0, \infty)$ is said to belong to the class Γ if there exists a function $v : \mathbb{R} \to (0, \infty)$ such that for all $\lambda \in \mathbb{R}$

$$\lim_{t\to\infty} \frac{f(t + \lambda v(t))}{f(t)} = e^\lambda; \tag{1.23}$$

we write $f \in \Gamma$ or $f \in \Gamma(v)$. The function v is called an *auxiliary function* for f.

The following functions satisfy (1.23) with the given auxiliary functions v:

$$f(t) = \exp(t^\alpha) \text{ for fixed } \alpha > 0 \quad \text{with } v(t) = \begin{cases} 1 & \text{for } t \le 0 \\ t^{1-\alpha}/\alpha & \text{for } t > 0, \end{cases}$$

$$f(t) = \exp(t \ln t) \quad \text{with } v(t) = \begin{cases} 1 & \text{for } t \le 1 \\ 1/\ln t & \text{for } t > 1, \end{cases}$$

$$f(t) = \exp(e^t) \quad \text{with } v(t) = e^{-t}.$$

We now give selected properties of functions in the class Γ.

- If $f \in \Gamma$, then $f \in \mathcal{RPV}(\infty)$.
- Relation (1.23) holds uniformly on each bounded interval.
- Any positive function z is an auxiliary function for f if and only if $z(t) \sim v(t)$ as $t \to \infty$.
- An auxiliary function v in (1.23) cannot grow so fast: $v(t)/t \to 0$ as $t \to \infty$.
- If $f \in \Pi(w)$ and $f(\infty) = \infty$, then $f^{\leftarrow} \in \Gamma(v)$ with $v(t) \sim w(f^{\leftarrow}(t))$ as $t \to \infty$.
- If $g \in \Gamma(v)$ and $g(\infty) = \infty$, then $g^{\leftarrow} \in \Pi(w)$ with $w(t) \sim v(g^{\leftarrow}(t))$ as $t \to \infty$.
- The statements $f \in \Gamma$ and

$$\lim_{t\to\infty} \frac{f(t) \int_0^t \int_0^s f(\tau)\,\mathrm{d}\tau\,\mathrm{d}s}{\left(\int_0^t f(s)\,\mathrm{d}s\right)^2} = 1$$

  are equivalent.
- If $f \in \Gamma$, then for all positive α

$$\lim_{t\to\infty} \frac{\int_0^t f^{\alpha}(s)\,\mathrm{d}s}{f^{\alpha-1}(t) \int_0^t f(s)\,\mathrm{d}s} = \frac{1}{\alpha}.$$

  Conversely, if a positive nondecreasing function satisfies this relation for some positive $\alpha \neq 1$, then $f \in \Gamma$.
- *(Representation)* The statements $f \in \Gamma$ and

$$f(t) = \exp\left\{\eta(t) + \int_0^t \frac{\psi(s)}{\varphi(s)}\,\mathrm{d}s\right\},$$

  where $\eta(t) \to c \in \mathbb{R}$, $\psi(t) \to 1$ as $t \to \infty$, φ is positive, absolutely continuous with $\varphi'(t) \to 0$ as $t \to \infty$, are equivalent. The auxiliary function of f may be taken as φ.
- *(Representation)* The statements $f \in \Gamma$ and

$$f(t) = \exp\left\{\eta(t) + \int_0^t \frac{1}{\omega(s)}\,\mathrm{d}s\right\},$$

  where $\eta(t) \to c \in \mathbb{R}$ as $t \to \infty$ and $\omega \in \mathcal{SN}$ ($\mathcal{SN}$ being defined below), are equivalent. The auxiliary function of f may be taken as ω.

- If $g \in \mathcal{SN}$ ($\mathcal{SN}$ being defined below), then f defined by
$$f(t) = \exp\left\{\int_0^t \frac{\mathrm{d}s}{g(s)}\right\}$$
satisfies $f \in \Gamma(g)$.
- If $f \in \Gamma$ has a nondecreasing positive derivative f', then $f' \in \Gamma$.
- If $f \in \Gamma(v)$, then
$$v(t) \sim \frac{\int_0^t f(s)\,\mathrm{d}s}{f(t)} \quad \text{as } t \to \infty.$$
Hence v can always be taken measurable.
- If $f \in \Gamma(v)$, then $v(t + \lambda v(t)) \sim v(t)$ as $t \to \infty$ uniformly on finite intervals of $\mathbb{R}$ (that is v is in the below defined class $\mathcal{SN}$).
- If $f \in \Gamma(v)$, then $\int_0^t f(s)\,\mathrm{d}s \in \Gamma(v)$.
- If $f_1 \in \mathcal{RV}(\vartheta)$ with $\vartheta > 0$ and $f_2 \in \Gamma$, then $f_1 \circ f_2 \in \Gamma$.
- If $f_1 \in \Gamma$ and $f_2' \in \mathcal{RV}(\vartheta)$ with $-1 < \vartheta < \infty$, then $f_1 \circ f_2 \in \Gamma$.
- If $f_1 \in \Gamma$ and $f_2' \in \Gamma$, then $f_1 \circ f_2 \in \Gamma$.
- For a nondecreasing function f, $f \in \Gamma \cup \mathcal{RV}$ if and only if
$$\lim_{t\to\infty} \frac{f(t) \int_0^t \int_0^s f(\tau)\,\mathrm{d}\tau\,\mathrm{d}s}{\left(\int_0^t f(s)\,\mathrm{d}s\right)^2} = C. \tag{1.24}$$
Under these conditions necessarily $C \in [1/2, 1]$. If $C = 1$, then $f \in \Gamma$. If $C < 1$, then $f \in \mathcal{RV}(1/(1-C) - 2)$.

We already know that $\Gamma \subseteq \mathcal{RPV}(\infty)$. For an example showing that the inclusion is strict see the proof of [14, Proposition 2.4.4].

In applications to differential equations we will also use the class Γ_- defined as follows. A function $f \in \Gamma_-(v)$ if $1/f \in \Gamma(v)$. Alternatively we can define: A function $f : (0,\infty) \to (0,\infty)$ is in the class $\Gamma(\sigma; v)$, σ being a real number and $v \in \mathcal{SN}$ (see below), if
$$\lim_{t\to\infty} \frac{f(t + \lambda v(t))}{f(t)} = \exp(\sigma\lambda)$$
for all real λ. Clearly, $\Gamma(1; v) = \Gamma(v)$ and $\Gamma(-1, v) = \Gamma_-(v)$. It holds:

$$\text{If } f' \in \Gamma(v), \text{ then } \lim_{t\to\infty} f(t) = A \text{ exists, } A - f(t) \sim v(t)f(t), \\ \text{and } A - f(t) \in \Gamma_-(v). \tag{1.25}$$

The class of auxiliary functions for functions in the class Γ is an interesting class in its own right since it can be used in either context as well.

Definition 1.9. A measurable function $f : \mathbb{R} \to (0,\infty)$ is *Beurling slowly varying* if

$$\lim_{t\to\infty} \frac{f(t+\lambda f(t))}{f(t)} = 1 \quad \text{for all } \lambda \in \mathbb{R}; \tag{1.26}$$

we write $f \in \mathcal{BSV}$. If (1.26) holds locally uniformly in λ, then f is called *self-neglecting*; we write $f \in \mathcal{SN}$.

Here are selected properties concerning the classes $\mathcal{BSV}$ and $\mathcal{SN}$.

- If $f \in \mathcal{BSV}$ is continuous, then $f \in \mathcal{SN}$.
- *(Representation)* It holds $f \in \mathcal{SN}$ if and only if it has the representation

  $$f(t) = \varphi(t) \int_0^t \psi(s)\,\mathrm{d}s,$$

  where $\lim_{t\to\infty} \varphi(t) = 1$ and ψ is continuous with $\lim_{t\to\infty} \psi(t) = 0$.
- If $f \in \mathcal{SN}$, then $\lim_{t\to\infty} f(t)/t = 0$.
- If $f \in \mathcal{BSV}$ is continuous, then there exists $g \in C^1$ such that $f(t) \sim g(t)$ and $g'(t) \to 0$ as $t \to \infty$.

Consider $f \in \mathcal{RV}(\vartheta)$, $\vartheta > -1$. Then

$$\frac{\int_0^t f(s)\,\mathrm{d}s}{tf(t)} \to \frac{1}{\vartheta+1} \quad \text{and} \quad \frac{\int_0^t \int_0^s f(\tau)\,\mathrm{d}\tau\,\mathrm{d}s}{t\int_0^t f(s)\,\mathrm{d}s} \to \frac{1}{\vartheta+2}$$

as $t \to \infty$, see the Karamata theorem and (1.24). By setting

$$h(t) = \int_0^t \int_0^s f(\tau)\,\mathrm{d}\tau\,\mathrm{d}s$$

and combining (1.1) with the above two relations we find that h satisfies the differential equation

$$h(t)h''(t) = \varphi(t)h'^2(t), \tag{1.27}$$

where $\varphi(t) \to (\vartheta+1)/(\vartheta+2)$. This leads to the idea to start from (1.27) and study the asymptotic behavior of nonnegative solutions of (1.27) under various conditions on φ, see Omey [132]. For instance, the following holds, cf. (1.24). Suppose that h is a nonnegative solution of (1.27) and $\varphi(t) \to C \in \mathbb{R} \cup \{\pm\infty\}$ as $t \to \infty$. If $C < 1$ or $C > 1$, then $h \in \mathcal{RV}(\beta)$ where $\beta = 1/(1-C)$; here $\beta = 0$ if $C = \pm\infty$. If $C = 1$, then $h \in \Gamma(d, v)$ where $v = |h/h'| \in \mathcal{SN}$ and $d = 1$ or $d = -1$ depending on the sign of h'. Second order behavior is also studied in Omey [132].

The behavior of a function $f \in \mathcal{BSV}$ is controlled by the function f itself. A similar class of functions controlled by another function z is defined as follows (see e.g. [131, 133]). Suppose that $z \in \mathcal{SN}$. A positive and measurable function f is controlled by z if for all real λ,

$$\frac{f(t+\lambda z(t))}{f(t)} \to 1$$

as $t \to \infty$; we write $f \in \mathcal{SC}(z)$. It holds: If $f' \in \mathcal{SC}(z)$, then for all real λ,

$$f(t+\lambda z(t)) - f(t) = (1+o(1))\lambda z(t) f'(t) \tag{1.28}$$

locally uniformly in λ. For other classes related to Γ see [133].

1.3 Some other related classes

1.3.1 Generalized regularly varying functions

The following type of extension of $\mathcal{RV}$ functions was introduced in [60]. Motivation was primarily for purposes of studying differential equations. Consider a continuously differentiable function ω which is positive and satisfies $\omega'(t) > 0$ for $t \in [b,\infty)$ and $\lim_{t\to\infty} \omega(t) = \infty$.

Definition 1.10. A measurable function $f : [a,\infty) \to (0,\infty)$ is called *regularly varying of index ϑ with respect to ω* if $f \circ \omega^{-1} \in \mathcal{RV}(\vartheta)$; we write $f \in \mathcal{RV}_\omega(\vartheta)$. If $\vartheta = 0$, then f is called *slowly varying with respect to ω*; we write $f \in \mathcal{SV}_\omega$.

The following selected properties of generalized $\mathcal{RV}$ functions are mostly immediate consequences of the properties of $\mathcal{RV}$ functions.

- $f \in \mathcal{RV}_\omega(\vartheta)$ if and only if $f(t) = \omega^\vartheta(t) L_\omega(t)$, where $L_\omega \in \mathcal{SV}_\omega$.
- If $L_\omega \in \mathcal{SV}_\omega$ and $\vartheta > 0$, then $\omega^\vartheta(t) L_\omega(t) \to \infty$, $\omega^{-\vartheta}(t) L_\omega(t) \to 0$ as $t \to \infty$.
- *(The representation theorem)*: $L_\omega \in \mathcal{SV}_\omega$ if and only if

$$L_\omega(t) = c(t) \exp\left\{\int_a^t \frac{\omega'(s)}{\omega(s)} h(s)\,\mathrm{d}s\right\}, \tag{1.29}$$

  $t \ge a$, for some $a > 0$, where c, h are measurable and $c(t) \to c \in (0,\infty)$, $h(t) \to 0$ as $t \to \infty$. If $c(t) \equiv c$ in (1.29), then L_ω is called *normalized slowly varying with respect to ω*; we write $L_\omega \in \mathcal{NSV}_\omega$. A generalized regularly varying function $f(t) = \omega^\vartheta(t) L_\omega(t)$ with $L_\omega \in \mathcal{NSV}_\omega$ is called *normalized regularly varying* of index ϑ with respect to ω; we write $f \in \mathcal{NRV}_\omega(\vartheta)$.

- Representation of a generalized $\mathcal{RV}$-function can alternatively be written as follows: $f \in \mathcal{RV}_\omega(\vartheta)$ if and only if

$$f(t) = c(t) \exp\left\{\int_a^t \frac{\omega'(s)}{\omega(s)} \delta(s)\,\mathrm{d}s\right\}, \tag{1.30}$$

  $t \ge a$, for some $a > 0$, where c, δ are measurable and $c(t) \to c \in (0,\infty)$, $\delta(t) \to \vartheta$ as $t \to \infty$.

- If $\omega(t) \sim K\varphi(t)$ as $t \to \infty$ for some constant $K > 0$, then
$$\mathcal{RV}_{\omega}(\vartheta) = \mathcal{RV}_{\varphi}(\vartheta)$$
for any $\vartheta \in \mathbb{R}$.
- It holds
$$\mathcal{RV}_{\omega^{\gamma}}(\vartheta) = \mathcal{RV}_{\omega}(\vartheta\gamma)$$
for any $\vartheta \in \mathbb{R}$ and $\gamma \in (0, \infty)$.
- There hold $\mathcal{RV}_{\mathrm{id}}(\vartheta) = \mathcal{RV}(\vartheta)$ and $\mathcal{NRV}_{\mathrm{id}}(\vartheta) = \mathcal{NRV}(\vartheta)$.

It would be of interest to observe that there exists a function which is slowly varying in the generalized sense but is not slowly varying in the sense of Karamata, so that, roughly speaking, the class of generalized Karamata functions is larger than that of classical Karamata functions. In fact, using the notation

$$\begin{aligned} &\exp_0 t = t, \;\; \exp_n t = \exp(\exp_{n-1} t), \quad n = 1, 2, \ldots, \\ &\ln_0 t = t, \;\; \ln_n t = \ln(\ln_{n-1} t), \quad n = 1, 2, \ldots, \end{aligned}$$

we define the functions $\phi_n(t)$ and $f_n(t)$ for $n \in \mathbb{Z}$ by

$$\phi_n(t) = \exp_n t, \;\; \phi_{-n}(t) = \log_n t, \quad n = 0, 1, 2, \ldots,$$

and

$$f_n(t) = 2 + \sin \phi_n(t), \quad n = 0, \pm 1, \pm 2, \ldots.$$

Since $\phi_n^{-1}(t) = \phi_{-n}(t)$ and $\phi_m \circ \phi_n(t) = \phi_{m+n}(t)$ for any $m, n \in \mathbb{Z}$, we have

$$f_n \circ \phi_m^{-1}(t) = f_{n-m}(t)$$

for any $m, n \in \mathbb{Z}$, from which, by taking into account the fact that

$$f_n(t) \in \mathcal{SV} \text{ for } n \le -2 \;\text{ and }\; f_n(t) \notin \mathcal{SV} \text{ for } n \ge -1,$$

we conclude that

$$f_n(t) \notin \mathcal{SV} \text{ and } f_n(t) \in \mathcal{SV}_{\phi_m} \;\text{ if } n \ge -1 \text{ and } m \ge n + 2.$$

Regular boundedness is generalized as follows.

Definition 1.11. A measurable function $f : [a, \infty) \to (0, \infty)$ is called *regularly bounded with respect to ω* if $f \circ \omega^{-1} \in \mathcal{RB}$; we write $f \in \mathcal{RB}_{\omega}$.

A function f belongs to $\mathcal{RB}_{\omega}$ if and only if it has the representation

$$f(t) = \exp\left\{\eta(t) + \int_a^t \frac{\omega'(s)}{\omega(s)} \xi(s)\,\mathrm{d}s\right\},$$

where η and ξ are bounded measurable functions on $[a, \infty)$.

1.3.2 Subexponential functions

Now we recal the concept of subexponential functions which have been shown useful in the study of asymptotic properties of differential equations, and are also somehow related to $\mathcal{RV}$ functions, see e.g. [5, 7].

The convolution of two appropriate functions f, g defined on $[0, \infty)$ is denoted, as usual, by

$$(f * g)(t) = \int_0^t f(t-s)g(s)\,\mathrm{d}s, \quad t \geq 0.$$

Definition 1.12. Let $f : [0, \infty) \to (0, \infty)$ be a continuous function. Then we say that f is *(positive) subexponential* if

$$\lim_{t\to\infty} \frac{(f * f)(t)}{f(t)} = 2\int_0^\infty f(s)\,\mathrm{d}s \tag{1.31}$$

(where we assume the convergence of the integral) and

$$\lim_{t\to\infty} \sup_{0\leq s\leq T} \left| \frac{f(t-s)}{f(t)} - 1 \right| = 0 \quad \text{for all } T > 0 \tag{1.32}$$

(i.e., $\lim_{t\to\infty} f(t-s)/f(t)$ uniformly for $0 \leq s \leq T$ for all $T > 0$).

The nomenclature subexponential is suggested by the fact that (1.32) implies that, for every $\varepsilon > 0$, $f(t)e^{\varepsilon t} \to \infty$ as $t \to \infty$, see e.g. [8]. It is also true that $\lim_{t\to\infty} f(t) = 0$. In the definition above, condition (1.31) can be replaced by

$$\lim_{T\to\infty} \limsup_{t\to\infty} \frac{1}{f(t)} \int_T^{t-T} f(t-s)f(s)\,\mathrm{d}s = 0$$

and this latter condition often proves to be useful in proofs.

The properties of subexponential functions have been extensively studied, for example, in [7, 8, 21]. Simple examples of subexponential functions are

$$\begin{aligned} f(t) &= (1+t)^{-\alpha} \text{ for } \alpha > 1, \\ f(t) &= e^{-(1+t)^{\alpha}} \text{ for } 0 < \alpha < 1, \\ f(t) &= e^{-t/\ln(t+2)}. \end{aligned}$$

The class of subexponential functions therefore includes a wide variety of functions exhibiting polynomial and slower-than-exponential decay: nor is the slower-than-exponential decay limited to a class of polynomially decaying functions. It is noted in [7] that the class of (positive) subexponential functions includes all continuous, integrable functions which are regularly varying at infinity. If $g \in \mathcal{RV}(\vartheta)$ with $\vartheta < -1$, g is subexponential.

1.3.3 Regular variation on various time scales

The concept of regular variation can be extended in such a way that it allows us to study asymptotic behavior of difference equations or q-difference equations or dynamic equations on time scales, see Section 6.2, Section 6.3, Section 6.4, respectively. Since this text is focused on differential equations, we will be very brief.

The concept of regularly sequences was introduced already by Karamata in 1930. In fact, two main approaches are known in the basic theory of regularly varying sequences: the approach due to Karamata [73], based on a definition that can be understood as a direct discrete counterpart of simple and elegant continuous definition, and the approach due to Galambos and Seneta [43], based on purely sequential definition.

Definition 1.13 (Karamata [73])**.** A positive sequence $\{y_k\}$, $k \in \mathbb{N}$, is said to be *regularly varying of index* ϑ, $\vartheta \in \mathbb{R}$, if

$$\lim_{k\to\infty} \frac{y_{[\lambda k]}}{y_k} = \lambda^{\vartheta} \text{ for all } \lambda > 0, \tag{1.33}$$

where $[u]$ denotes the integer part of u.

Definition 1.14 (Galambos and Seneta [43])**.** A positive sequence $\{y_k\}$, $k \in \mathbb{N}$, is said to be *regularly varying of index* ϑ, $\vartheta \in \mathbb{R}$, if there is a positive sequence $\{\alpha_k\}$ satisfying

$$\lim_{k\to\infty} \frac{y_k}{\alpha_k} = C, \quad \lim_{k\to\infty} k\left(1 - \frac{\alpha_{k-1}}{\alpha_k}\right) = \vartheta, \tag{1.34}$$

C being a positive constant.

If $\varrho = 0$ in Definition 1.13 or 1.14, then $\{y_k\}$ is said to be *slowly varying*.

In Bojanić, Seneta [17], it was shown that Definition 1.13 is equivalent to Definition 1.14.

In Matucci, Řehák [119] the authors are interested in applying regularly varying sequences to certain second order difference equations. For this purpose a slight modification (in the equivalent sense) of Definition 1.14 is proposed there; the latter condition in (1.34) is replaced by

$$\lim_{k\to\infty} \frac{k\Delta\alpha_k}{\alpha_k} = \vartheta.$$

In Bojanić, Seneta [17] and Galambos, Seneta [43], the so-called embedding theorem was established (and the converse result holds as well):

Theorem 1.6. *If $\{y_k\}$ is a regularly varying sequence, then the function R (of a real variable), defined by $R(t) = y_{[t]}$, is regularly varying.*

Such a result makes it then possible to apply the continuous theory to the theory of regularly varying sequences. However, the development of a discrete theory, analogous to the continuous one, is not generally close, and sometimes far from a simple imitation of arguments for regularly varying functions, as noticed and demonstrated in Bojanić, Seneta [17]. Simply, the embedding theorem is just one of powerful tools, but sometimes it is not immediate that from a continuous results its discrete counterpart is easily obtained thanks to the embedding; sometimes it is even not possible to use this tool and the discrete theory requires a specific approach, different from the continuous one.

For further properties of regularly varying sequences and some other related useful concepts (such as rapidly varying sequences) see e.g. [17, 25, 26, 43].

References concerning applications of regular variation in difference equations are given in Section 6.2.

In Řehák [141], the concept of regularly varying functions on time scales (or measure chains) was introduced; the primary purpose was to investigate dynamic equations on time scales.

Recall that the calculus on time scales (or, more generally, on measure chains) deals essentially with functions defined on nonempty closed subsets of $\mathbb{R}$, see Hilger's initiating work [55] and the monograph [16] by Bohner, Peterson. Hence, it unifies and extends usual calculus and quantum (q- or h-) calculi.

A theory of regular variation on time scales offers something more than the embedding result, and has the following advantages: Once there is proved a result on a general time scale, it automatically holds for the continuous and the discrete case, without any other effort. Moreover, at the same time, the theory works also on other time scales which may be different from the "classical" ones.

A time scale $\mathbb{T}$ is assumed to be unbounded above. The following definition is motivated by a modification of the purely sequential criterion mentioned above.

Definition 1.15. A measurable function $f : \mathbb{T} \to (0, \infty)$ is said to be *regularly varying of index* ϑ, $\vartheta \in \mathbb{R}$, if there exists a positive rd-continuously delta differentiable function α satisfying

$$f(t) \sim C\alpha(t) \quad \text{and} \quad \lim_{t\to\infty} \frac{t\alpha^{\Delta}(t)}{\alpha(t)} = \vartheta,$$

C being a positive constant. If $\vartheta = 0$, then f is said to be *slowly varying*.

In Řehák, Vítovec [153], a Karamata type definition for $\mathcal{RV}$ functions on $\mathbb{T}$ is introduced and an embedding theorem is proved. Note that conditions posed on the behavior of the graininess $\mu(t)$ of a time scale $\mathbb{T}$ plays a crucial role in the theory. In particular we suggest to distinguish three cases: (a) The graininess satisfies the condition $\mu(t) = o(t)$ as $t \to \infty$. Then we obtain a continuous like theory which unifies the above discrete and continuous theories. (b) The case where $\mu(t) = Ct$ with $C \in (0, \infty)$ leads to the q-case, which is discussed below. (c) Other cases — in particular when the graininess is "too large" or a "combination of large and small" — give no reasonable theory of regular variation in a certain sense.

Applications of this theory (especially when $\mu(t) = o(t)$) to dynamic equations on time scales can be found in the works listed in Section 6.3.

The concept of the so-called q-regularly varying functions was introduced in Řehák, Vítovec [153]. Let $q^{\mathbb{N}_0}$ denote the q-uniform lattice $q^{\mathbb{N}_0} = \{q^k : k \in \mathbb{N}_0\}, q > 1$. Let $D_q f$ denote the Jackson derivative of f. Further we set $[a]_q = (q^a - 1)/(q - 1)$ for $a \in \mathbb{R}$. Quantum version of regular variation of $f : q^{\mathbb{N}_0} \to (0, \infty)$ can be defined in accordance with Definition 1.15 as follows.

Definition 1.16. A function $f : q^{\mathbb{N}_0} \to (0, \infty)$ is said to be *q-regularly varying of index* ϑ, $\vartheta \in \mathbb{R}$, if there exists a function $\beta : q^{\mathbb{N}_0} \to (0, \infty)$ satisfying

$$f(t) \sim C\beta(t), \text{ and } \lim_{t\to\infty} \frac{tD_q\beta(t)}{\beta(t)} = [\vartheta]_q,$$

C being a positive constant. If $\vartheta = 0$, then f is said to be *q-slowly varying*.

However, as was shown in the quoted paper, thanks to the structure of $q^{\mathbb{N}_0}$, we are able to find much simpler (and still equivalent) characterization which has no analogue in the classical continuous or the discrete case. Such a simplification is possible since q-regular variation can be characterized in terms of relations between $f(t)$ and $f(qt)$, which is natural for discrete q-calculus, in contrast to other settings. In particular, for $f : q^{\mathbb{N}_0} \to (0, \infty)$ we have

$$f \text{ is } q\text{-regularly varying of index } \vartheta \iff \lim_{t\to\infty} \frac{f(qt)}{f(t)} = q^{\vartheta}.$$

For further properties of q-regularly varying functions (and other related functions such as q-rapidly varying ones) see [142] and [154]. Note that in [145], certain generalization of q-regular variation was introduced which somehow involves also q-rapid variation or q-hypergeometric functions.

Applications of this theory to q-difference equations can be found in the works listed in Section 6.4.

1.3.4 Hardy field

The so-called *logarithmico-exponential function* is defined as a real-valued function defined on $[a, \infty)$ by

$$\begin{cases} \text{a finite combination of the ordinary symbols } (+, -, \cdot, /, \sqrt[n]{\ }) \\ \text{and the functional symbols } \ln(\cdot), \exp(\cdot), \text{ operating} \\ \text{on the variable } x \text{ and on real constants,} \end{cases}$$

see Hardy [52, III.2].

More generally, see Bourbaki [18], a *Hardy field* is

$$\begin{cases}\text{a set of germs of real-valued functions on } [a, \infty) \text{ that is closed}\\ \text{under differentiation and that form a field under the usual}\\ \text{addition and multiplication of germs,}\end{cases}$$

see also Rosenlicht [155]. Loosely speaking, Hardy fields are the natural domain of asymptotic analysis, where all rules hold without qualifying conditions.

In Hardy's own words [52, V.6]: "No function has yet presented itself in analysis the laws of whose increase, in so far as they can be stated at all, cannot be stated, so to stay, in logarithmico-exponential terms." This statement of Hardy was basically influenced by the fact that the arithmetic functions occurring in the number theory having often very complicated structure and for which he expected "would give rise to genuinely new modes of increase," so far obey the log-exp laws of increase.

That indicates a possible significance of the results in this treatise. For any logarithmico-exponential function f (or any element of Hardy fields) together with the derivatives, is ultimately continuous and monotonic, of constant sign and $\lim_{x\to\infty} f(x)$ exists as a finite or infinite one. On the other hand, as we could see above, a slowly varying function may somehow oscillate, even infinitely. As some of the results in the next chapter show, solutions of differential solutions (for instance, of a second-order (half-)linear one) may behave as $\mathcal{SV}$ functions. Therefore, the solutions of such a (simple) equation may exhibit a "genuinely new mode of increase." To support our point we emphasize here that no hypothesis of some of the theorems which lead to the above statement about solutions is related to regular variation.

For completeness, note that Rosenlicht in [155] looks for asymptotic formulas for solutions to the linear differential equation $y'' = p(t)y$ with the help of the theory of Hardy fields. It is assumed that $p(t) \to \infty$ as $t \to \infty$ and p belongs to a Hardy field. In fact, the problem is transferred to the investigation of the associated Riccati equation.

Chapter 2

Linear second order differential equations

Although this text is focused primarily on applications of the theory of regular variation to nonlinear differential equations, we start with linear equations. The objective of this chapter is multiple: comparison purposes, reference purposes, but also other ones. Indeed, many of the results which we present here will be later generalized to a nonlinear case. Second, some of the results for linear equations will be utilized in the study of nonlinear equations in various ways. Third, some of the statements may serve as a motivation for an attempt to extend them (in particular, to the half-linear case or to the nearly linear case). Moreover, our survey includes also the results that appeared after Marić's book [105] and we point out new relations among various results, and revise some of them.

Since we want primarily to deal with nonlinear equations, we present the results in this chapter without proofs and give only comments; they include the main ideas in some instances.

Perhaps the first paper where linear differential equations were studied in the framework of regular variation is [130] (Omey, 1981). Note that a connection of regular variation with nonlinear equations was shown much earlier by Avakumović in 1947, see [9] and Section 4.2.

We consider linear second order equation

$$y'' + p(t)y = 0, \tag{2.1}$$

where p is a continuous function on $[a, \infty)$. Many of the results in this chapter were obtained by Marić, Tomić, and their collaborators. Another important authors in this direction are Howard, Geluk, Grimm, Hall, Omey, and Radašin – this concerns the research up to the year 2000. For more information see the monograph [105] by Marić. Recent results for linear equations presented here are mainly due to Jaroš, Kusano, Marić, and Řehák.

2.1 Karamata solutions; coefficient with sign condition

Instead of (2.1) it is more convenient to consider now

$$y'' - p(t)y = 0. \tag{2.2}$$

In the following theorem, the sign condition on p is assumed. Note that under this condition the equation is nonoscillatory (by the Sturm comparison theorem). Without loss of generality, we analyze only positive solutions.

Note that although the statements refer to regular or rapid variation, none of the hypotheses of the theorem requires this notion.

Theorem 2.1 (Marić, Tomić [105, 114]). *Let p be positive.*

(i) Equation (2.2) *has a fundamental set of solutions $y_1(t) = L_1(t) \in \mathcal{SV}$, $y_2(t) = tL_2(t) \in \mathcal{RV}(1)$ if and only if*

$$\lim_{t\to\infty} t \int_t^\infty p(s)\,\mathrm{d}s = 0.$$

Moreover, $L_1, L_2 \in \mathcal{NSV}$ with $L_2(t) \sim 1/L_1(t)$. All positive decreasing solutions of (2.1) *are in $\mathcal{NSV}$ and all positive increasing solutions are in $\mathcal{NRV}(1)$.*

(ii) Equation (2.2) *has a fundamental set of solutions $y_1(t) = t^{\vartheta_1}L_1(t) \in \mathcal{RV}(\vartheta_1)$, $y_2(t) = t^{\vartheta_2}L_2(t) \in \mathcal{RV}(\vartheta_2)$ if and only if*

$$\lim_{t\to\infty} t \int_t^\infty p(s)\,\mathrm{d}s = C,$$

where $\vartheta_1 < \vartheta_2$ are the roots of the equation $\vartheta^2 - \vartheta - C = 0$. Moreover, $L_1, L_2 \in \mathcal{NSV}$ with $L_2(t) \sim 1/((1-\vartheta_1)L_1(t))$. All positive decreasing solutions of (2.1) *are in $\mathcal{NRV}(\vartheta_1)$ and all positive increasing solutions are in $\mathcal{NRV}(\vartheta_2)$.*

(iii) Equation (2.2) *has a fundamental set of solutions $y_1 \in \mathcal{RPV}(-\infty)$, $y_2 \in \mathcal{RPV}(\infty)$ if and only if for each $\lambda > 1$*

$$\lim_{t\to\infty} t \int_t^{\lambda t} p(s)\,\mathrm{d}s = \infty.$$

Moreover, all positive decreasing solutions of (2.1) *are in $\mathcal{RPV}(-\infty)$ and all positive increasing solutions are in $\mathcal{RPV}(\infty)$.*

It would be of interest whether the integrals in conditions in all three parts of the theorem were the same. To make a conclusion, the following observations are important (see Grimm, Hall [49]): $t\int_t^\infty p(s)\,\mathrm{d}s \to 0$ as $t \to \infty$ if and only if, for each $\lambda > 1$, $t\int_t^{\lambda t} p(s)\,\mathrm{d}s \to 0$ as $t \to \infty$. Also, $t\int_t^\infty p(s)\,\mathrm{d}s \to C > 0$ as $t \to \infty$ if and only if, for each $\lambda > 1$, $t\int_t^{\lambda t} p(s)\,\mathrm{d}s \to C(\lambda-1)/\lambda$ as $t \to \infty$. Thus, the interval of integration in all three conditions may be taken to be $(t, \lambda t)$. On the other hand, these are easier to verify for the interval (t, ∞) as it appears in the first of them. However in the third condition the interval $(t, \lambda t)$ cannot be replaced by (t, ∞) even when the integral $\int_t^\infty p(s)\,\mathrm{d}s$ converges. For, the condition $t\int_t^\infty p(s)\,\mathrm{d}s \to \infty$ as $t \to \infty$ does not necessarily imply even that $\lim_{t\to\infty} t\int_t^{\lambda t} p(s)\,\mathrm{d}s$ exists for all $\lambda > 1$. A counter-example is presented in Grimm, Hall [49].

From the previous theorem we get the following statement. Recall that by Karamata functions we mean $\mathcal{SV}$ or $\mathcal{RV}$ or $\mathcal{RPV}$ functions.

Corollary 2.1. *All (eventually positive) solutions are Karamata functions if and only if there exists for each $\lambda > 1$,*

$$\lim_{t\to\infty} t\int_t^{\lambda t} p(s)\,\mathrm{d}s$$

as a finite or infinite one.

Marić and Tomić in [113] proved a similar result under a more restrictive assumption on p, namely that the limit $\lim_{t\to\infty} t^2 p(t)$ exists. This is a sufficient condition for (2.1) to be in the Karamata class. More precisely, we have the following

Corollary 2.2 (Marić, Tomić [113])**.** *If $\lim_{t\to\infty} t^2 p(t) = A$, then all decreasing solutions of* (2.2) *are slowly or rapidly or regularly varying functions with the index $\vartheta = (1 - \sqrt{1+4A})/2$ in the latter case, according to as $A = 0$, $A = \infty$, $A \in (0,\infty)$, respectively.*

The second linearly independent solution can be treated as in the general case. That is, with the use of the reduction of order formula in the $\mathcal{SV}$ and $\mathcal{RV}$ cases, and directly in the $\mathcal{RPV}$ case.

Such a result was in essence first discovered by Omey in [130, Theorem 2.1] (with an additional assumption) and formulated for solutions tending to infinity. This excludes the case $A = 0$ or $\vartheta = 0$, since $\mathcal{SV}$ solutions cannot cannot increase. Thus the "trichotomy" character of the result is lost in such formulation.

An important feature of Corollary 2.2 consists in opening various possibilities of a subtle use of classes $\Pi, \mathcal{BSV}$ and Γ in further analysis of solutions under consideration. Thus, by specifying the way in which $t^2 f(t)$ tends to the finite limit A, Geluk proved in [45] some refinements of Corollary 2.2, see Subsection 2.6.1. In that he makes use of the class Π. Analogous results for the case $A = \infty$ are obtained in [130, 131] by Omey using classes $\mathcal{BSV}$ and Γ, see Subsection 2.6.1. Also some of these results were generalized to the half-linear case and improved even in the linear case, see Section 3.6. In Section 5.5 we discuss this type of results for nearly linear equations.

2.2 Karamata solutions; coefficient with no sign condition

Now consider equation (2.1) with no sign condition on p. Such an equation may have oscillatory solutions. However, since we are interested in solutions belonging to Karamata class whose elements are positive, only nonoscillatory solutions have to be considered. Observe that expressions in conditions guaranteeing regular variation or regular boundedness resemble expressions which appear in the well known Hille-Nehari type (non)oscillation criteria.

In contrast to the case when $p(t) < 0$ in (2.1) and eventually positive solutions always exist, here we have to establish the existence of nonoscillatory solutions first. This can be achieved either by proving it ab ovo, in fact simultaneously with the regularity like in subsequent Theorems 2.3 and 2.4, or by applying the following auxiliary result. In addition to the methods used by Howard, Marić, Radašin [57, 56], there exists also a different approach, based on the Banach contraction mapping principle, see Jaroš, Kusano [58] — both, principal and nonprincipal solutions can be directly constructed. Results in Section 3.2 can be understood as a half-linear extension of this approach.

The following result very well suits our needs. The proof uses the method of successive approximation and a variant of the Riccati technique can be revealed in it.

Proposition 2.1 (Howard, Marić, Radašin [57, 105])**.** *Put*

$$P(t) = \int_t^\infty p(s)\,\mathrm{d}s. \tag{2.3}$$

If there exists a positive continuous function h with $h(t) \to 0$ as $t \to \infty$, and such that for $t \ge t_0$, $|P(t)| \le h(t)$, $\int_t^\infty h^2(t)\,\mathrm{d}t \le Ah(t)$ with $0 < A < 1/4$, then (2.1) *is nonoscillatory and there exists a solution of the form*

$$y(t) = \exp\left\{\int_a^t (P(s) - Z(s))\,\mathrm{d}s\right\}. \tag{2.4}$$

Here Z is a solution of the integral equation

$$Z(t) = -\int_t^\infty (Z(s) - P(s))^2 \mathrm{d}s \tag{2.5}$$

satisfying $Z(t) = O(h(t))$ as $t \to \infty$.

The previous proposition plays a key role in the proof of the following generalization of Theorem 2.1-(i).

Theorem 2.2 (Howard, Marić [56, 105])**.** *Equation* (2.1) *has a fundamental set of solutions $y_1(t) = L_1(t) \in \mathcal{SV}$, $y_2(t) = tL_2(t) \in \mathcal{RV}(1)$ if and only if*

$$\lim_{t\to\infty} t \int_t^\infty p(s)\,\mathrm{d}s = 0. \tag{2.6}$$

Moreover, $L_1, L_2 \in \mathcal{NSV}$ with $L_2(t) \sim 1/L_1(t)$ as $t \to \infty$.

In the next result, Proposition 2.1 cannot be applied, and one has to prove regularity and thus nonoscillation directly.

Theorem 2.3 (Howard, Marić [56, 105]). *Let $A \in (-\infty, 1/4)$, $A \neq 0$, and let $\vartheta_1 < \vartheta_2$ be the roots of the equation $\vartheta^2 - \vartheta + A = 0$. Equation* (2.1) *has a fundamental set of solutions $y_1(t) = t^{\vartheta_1} L_1(t) \in \mathcal{RV}(\vartheta_1)$, $y_2(t) = t^{\vartheta_2} L_2(t) \in \mathcal{RV}(\vartheta_2)$ if and only if*

$$\lim_{t\to\infty} t \int_t^\infty p(s)\,\mathrm{d}s = A. \tag{2.7}$$

Moreover, $L_1, L_2 \in \mathcal{NSV}$ with $L_2(t) \sim 1/((1-\vartheta_1)L_1(t))$ as $t \to \infty$.

The next theorem deals with the border case when $A = 1/4$ in (2.7). Note that in general no conclusion concerning oscillation or nonoscillation of (2.1) can be drawn. Also in this result, Proposition 2.1 cannot be applied.

Theorem 2.4 (Howard, Marić [56, 105]). *Let $A = 1/4$ and suppose that the integral $\int^\infty \frac{\phi(s)}{s}\,\mathrm{d}s$ converges. Put*

$$\phi(t) := t \int_t^\infty p(s)\,\mathrm{d}s - A \quad \textit{and} \quad \psi(t) := \int_t^\infty \frac{\phi(s)}{s}\,\mathrm{d}s.$$

Assume

$$\int^\infty \frac{\psi(s)}{s}\,\mathrm{d}s < \infty.$$

Then equation (2.1) *has a fundamental set of solutions $y_1(t) = \sqrt{t} L_1(t) \in \mathcal{RV}(1/2)$, $y_2(t) = \sqrt{t}\ln t L_2(t)$ if and only if* (2.7) *holds. Moreover, $L_1, L_2 \in \mathcal{NSV}$, tend to constants and $L_2(t) \sim 1/L_1(t)$ as $t \to \infty$.*

In the above theorems the existence of the limit of $t\int_t^\infty p(s)\,\mathrm{d}s$ is required. If one relaxes that request to a condition of Hille-Nehari type, then we get regular boundedness. Proposition 2.1 finds application in the proof.

Theorem 2.5 (Howard, Marić [56, 105]). *If, for large t,*

$$\left| t \int_t^\infty p(s)\,\mathrm{d}s \right| \le A < \frac{1}{4},$$

then all (eventually positive) solutions of (2.1) *are in $\mathcal{RB}$.*

2.3 Generalization and self-adjoint equation

It is natural to ask whether the results of the previous section can be extended to the more general equation

$$y'' + g(t)y' + h(t)y = 0 \tag{2.8}$$

or to the equation in the self-adjoint form

$$(r(t)y')' + p(t)y = 0. \tag{2.9}$$

The next result is an easy consequence of Theorem 2.3, it is based on a suitable transformation.

Theorem 2.6 (Marić, Tomić [105, 116]). *Let $A, B \in \mathbb{R}$ be such that*

$$\frac{B}{2} - \frac{B^2}{4} + A < \frac{1}{4} \tag{2.10}$$

and let $\alpha_1 < \alpha_2$ be the roots of the equation $\alpha^2 - \alpha + \gamma = 0$ with $\gamma = B/2 - B^2/4 + A$. Further, let h be continuous and g continuously differentiable on $[t_0, \infty)$ and such that $tg(t) \to B$ as $t \to \infty$. Then there exist two linearly independent regularly varying solutions y_1, y_2 of (2.8) *of the form $y_i(t) = t^{\alpha_i - B/2} L_i(t)$, $i = 1, 2$, if and only if*

$$\lim_{t\to\infty} t \int_t^\infty h(s)\,\mathrm{d}s = A.$$

Here $L_1, L_2 \in \mathcal{NSV}$ are such that

$$L_2(t) \sim \frac{1}{(1 - 2\alpha_1)L_1(t)} \exp\left\{\int_a^t \frac{\varepsilon(s)}{s}\,\mathrm{d}s\right\}$$

with some $\varepsilon(t) \to 0$ as $t \to \infty$.

Condition (2.10) excludes the case $\alpha_1 = 0$. Observe that $A = B = 0$ implies $\alpha_1 = 0, \alpha_2 = 1$ so that $y_1 \in \mathcal{NSV}$ and $y_2 \in \mathcal{NRV}(1)$.

From the previous theorem, taking $g(t) = r'(t)/r(t)$, $h(t) = p(t)/r(t)$, we get the result for self-adjoint equation (2.9).

Corollary 2.3. *Let A, B, α_1, α_2 be as in Theorem 2.6. Let p be continuous, r positive and twice continuously differentiable on $[t_0, \infty)$, and such that*

$$\frac{tr'(t)}{r(t)} \to B \quad \text{as } t \to \infty. \tag{2.11}$$

Then (2.9) *has two $\mathcal{RV}$ solutions y_i having the same form as in Theorem 2.6 if and only if*

$$\lim_{t\to\infty} t \int_t^\infty \frac{p(s)}{r(s)}\,\mathrm{d}s = A.$$

Using the method of Theorem 2.1, Grimm and Hall [49] obtained similar result but for $p(t) < 0$ only, requiring of p to increase and tend to a (finite) limit as $t \to \infty$. This is more restrictive than (2.11) but, as a compensation, r may be only once continuously differentiable. They also generalized Theorem 2.6-(iii) using the same method:

Theorem 2.7 (Grimm and Hall [49]). *Let y be a decreasing solution of* (2.9)*, $p(t) < 0$, and $r \in \mathcal{NSV}$ be nondecreasing with $r(t) \to 1$ as $t \to \infty$. Then $y \in \mathcal{RPV}$ if and only if for each $\lambda > 1$, $t \int_t^{\lambda t} p(s)\,\mathrm{d}s \to \infty$ as $t \to \infty$.*

Jaroš and Kusano in [60] used a different approach to investigation of $\mathcal{RV}$-type solutions to (2.9). It utilizes the concept of generalized regular variation, see Subsection 1.3.1, which was introduced for this purpose in [60]. The contraction mapping theorem plays also a role. These results were generalized later to the half-linear case, see Section 3.4. For comparison purposes we give the results for linear equation here. Note however that some differences may occur. For instance, in the linear case, once we know the behavior of one solution, then — in many cases — it is easy to examine behavior of a linearly independent solutions. Further, once we have a fundamental set of $\mathcal{RB}$ solutions, it is almost immediate (because of linear combinations) to show that all positive solutions are $\mathcal{RB}$. On the other hand, the solution space for half-linear equations is not linear and the reduction of order formula cannot be used.

No differentiability condition on r and no sign condition on p is assumed. Essentially, we require p, r to be continuous with $r > 0$. The two cases are distinguished; this is quite usual when dealing with equations of the form (2.9):

$$\int_a^\infty \frac{\mathrm{d}s}{r(s)}\,\mathrm{d}s = \infty, \tag{2.12}$$

and

$$\int_a^\infty \frac{\mathrm{d}s}{r(s)}\,\mathrm{d}s < \infty. \tag{2.13}$$

We denote $R(t) = \int_a^t \frac{\mathrm{d}s}{r(s)}\,\mathrm{d}s$ if (2.12) holds and $\tilde{R}(t) = \int_t^\infty \frac{\mathrm{d}s}{r(s)}\,\mathrm{d}s$ if (2.13) holds.

In the next theorem we assume that p is integrable on $[a, \infty)$.

Theorem 2.8 (Jaroš, Kusano [60])**.** *Let* (2.12) *hold.*

(i) Let $A \in (-\infty, 1/4)$ and denote by ϑ_1, ϑ_2, $\vartheta_1 < \vartheta_2$, the real roots of the quadratic equation $\vartheta^2 - \vartheta + A = 0$. Equation (2.9) *has a fundamental set of solutions $\{y_1, y_2\}$ such that $y_i \in \mathcal{NRV}_R(\vartheta_i)$, $i = 1, 2$, if and only if*

$$\lim_{t\to\infty} R(t) \int_t^\infty p(s)\,\mathrm{d}s = A.$$

(ii) Assume

$$\lim_{t\to\infty} R(t) \int_t^\infty p(s)\,\mathrm{d}s = 1/4.$$

Put

$$\phi(t) = R(t) \int_t^\infty p(s)\,\mathrm{d}s - \frac{1}{4}$$

and suppose that

$$\int^\infty \frac{|\phi(s)|}{r(s)R(s)}\,\mathrm{d}s < \infty \quad \textit{and} \quad \int^\infty \frac{\psi(s)}{r(s)R(s)}\,\mathrm{d}s < \infty,$$

where $\psi(t) = \int_t^\infty \frac{|\phi(s)|}{r(s)R(s)}\,\mathrm{d}s$. Then equation (2.9) *possesses a fundamental set of solutions $\{y_1, y_2\}$ such that $y_i \in \mathcal{NRV}_R(1/2)$, $i = 1, 2$, and $y_1(t) = \sqrt{R(t)}L_1(t)$, $y_2(t) =$*

$\sqrt{R(t)}L_2(t)\ln R(t)$, *where* $L_1, L_2 \in \mathcal{NSV}_R$ *and* $\lim_{t\to\infty} L_i(t) = C_i \in (0,\infty)$, $i = 1,2$, *with* $C_1C_2 = 1$.

(iii) If

$$-\frac{1}{4} < \liminf_{t\to\infty} R(t)\int_t^\infty p(s)\,\mathrm{d}s \le \limsup_{t\to\infty} R(t)\int_t^\infty p(s)\,\mathrm{d}s < \frac{1}{4},$$

then (2.9) *is nonoscillatory and all its eventually positive solutions are in* $\mathcal{RB}_R$.

The next theorem is a complement of the previous one, in the sense of condition (2.13). We assume that $\tilde{R}^2(t)p(t)$ is integrable on $[a,\infty)$.

Theorem 2.9 (Jaroš, Kusano [60])**.** *Let* (2.13) *hold.*

(i) Let $B \in (-\infty, 1/4)$ *and denote by* μ_1, μ_2, $\mu_1 < \mu_2$, *the real roots of the quadratic equation* $\mu^2 - \mu + B = 0$. *Equation* (2.9) *has a fundamental set of solutions* $\{y_1, y_2\}$ *such that* $y_i \in \mathcal{NRV}_{1/\tilde{R}}(\mu_i)$, $i = 1,2$, *if and only if*

$$\lim_{t\to\infty} \frac{1}{\tilde{R}(t)}\int_t^\infty \tilde{R}^2(s)p(s)\,\mathrm{d}s = B.$$

(ii) Assume

$$\lim_{t\to\infty} \frac{1}{\tilde{R}(t)}\int_t^\infty \tilde{R}^2(s)p(s)\,\mathrm{d}s = 1/4.$$

Put

$$\phi(t) = \frac{1}{\tilde{R}(t)}\int_t^\infty \tilde{R}^2(s)p(s)\,\mathrm{d}s - \frac{1}{4}$$

and suppose that

$$\int^\infty \frac{|\phi(s)|}{r(s)\tilde{R}(s)}\,\mathrm{d}s < \infty \quad \text{and} \quad \int^\infty \frac{\psi(s)}{r(s)\tilde{R}(s)}\,\mathrm{d}s < \infty,$$

where $\psi(t) = \int_t^\infty \frac{|\phi(s)|}{r(s)\tilde{R}(s)}\,\mathrm{d}s$. *Then equation* (2.9) *possesses a fundamental set of solutions* $\{y_1, y_2\}$ *such that* $y_i \in \mathcal{NRV}_{1/\tilde{R}}(-1/2)$, $i = 1,2$, *and*

$$y_1(t) = \sqrt{\tilde{R}(t)}L_1(t), \quad y_2(t) = \sqrt{\tilde{R}(t)}L_2(t)\ln\frac{1}{\tilde{R}(t)},$$

where $L_1, L_2 \in \mathcal{NSV}_{1/\tilde{R}}$ *and* $\lim_{t\to\infty} L_i(t) = C_i \in (0,\infty)$, $i = 1,2$, *with* $C_1C_2 = 1$.

(iii) If

$$-\frac{1}{4} < \liminf_{t\to\infty} \frac{1}{\tilde{R}(t)}\int_t^\infty \tilde{R}^2(s)p(s)\,\mathrm{d}s \le \limsup_{t\to\infty} \frac{1}{\tilde{R}(t)}\int_t^\infty \tilde{R}^2(s)p(s)\,\mathrm{d}s < \frac{1}{4},$$

then (2.9) *is nonoscillatory and all its eventually positive solutions are in* $\mathcal{RB}_{1/\tilde{R}}$.

2.4 Linear differential equations having regularly varying solutions

Kusano and Marić in [94] deals with the question whether for any distinct real constants ϑ_1 and ϑ_2 there exists a differential equation of the form (2.9) which possesses a pair of solutions $y_i \in \mathcal{RV}(\vartheta_i)$, $i = 1, 2$. The problem was solved for more general, half-linear, equation. Here we formulate the linear version. The general statement along with the proof is presented in Section 3.7. The function ω which appears in the theorem is assumed to satisfy conditions from the definition of regularly varying functions with respect to ω, see Definition 1.10.

Theorem 2.10 (Kusano and Marić [94]). *Let ϑ_1 and ϑ_2 be any given real constants such that $|\vartheta_1| \neq |\vartheta_2|$.*

(i) Suppose that r satisfies $1/r(t) \sim K\omega^{\vartheta_1+\vartheta_2-1}(t)\omega'(t)$ as $t \to \infty$ for some positive constant K. Let $\vartheta_1 + \vartheta_2 > 0$ and p be conditionally integrable on $[a, \infty)$. Then equation (2.9) *possesses a fundamental set of solutions $y_i \in \mathcal{NRV}_\omega(\vartheta_i)$, $i = 1, 2$, if and only if*

$$\lim_{t\to\infty} K\omega^{\vartheta_1+\vartheta_2}(t) \int_t^\infty p(s)\,\mathrm{d}s = \frac{\vartheta_1\vartheta_2}{\vartheta_1 + \vartheta_2}.$$

(ii) Suppose that $1/r(t) = K\omega^{\vartheta_1+\vartheta_2-1}(t)\omega'(t)$ for some positive constant K. Let $\vartheta_1 + \vartheta_2 < 0$ and $\omega^{2(\vartheta_1+\vartheta_2)}p$ be conditionally integrable on $[a, \infty)$. Then equation (2.9) *possesses a fundamental set of solutions $y_i \in \mathcal{NRV}_\omega(\vartheta_i)$, $i = 1, 2$, if and only if*

$$\lim_{t\to\infty} K\omega^{-(\vartheta_1+\vartheta_2)}(t) \int_t^\infty \omega^{2(\vartheta_1+\vartheta_2)}(s)p(s)\,\mathrm{d}s = -\frac{\vartheta_1\vartheta_2}{\vartheta_1 + \vartheta_2}.$$

2.5 Regularly varying solutions of Friedmann equations

Mijajlović, Pejović, Šegan, and Damljanović in [125] applied the theory of $\mathcal{RV}$ functions to the asymptotical analysis at infinity of solutions of Friedmann cosmological equations. Their analysis is strongly based on Theorem 2.2 and Theorem 2.3.

Let us consider Friedmann equations

$$\left(\frac{u'}{u}\right)^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{u^2} \quad \text{(Friedmann equation)} \tag{2.14}$$

$$\frac{u''}{u} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right) \quad \text{(acceleration equation)} \tag{2.15}$$

that describe the evolution of the expansion scale factor $u(t)$ of the universe. Here, $p = p(t)$ is the energy pressure in the universe, $\rho = \rho(t)$ is the density of matter in the universe, k is the space curvature, G is the gravitational constant and c is the speed of light. The variable t represents the cosmic time.

In [125], it is established a necessary and sufficient condition for solutions that satisfy the generalized power law expressed as $u(t) = t^{\vartheta}L(t)$, where $L \in \mathcal{SV}$. The analysis is strongly based on Theorem 2.2 and Theorem 2.3. For this reason it is introduced a new parameter $\mu(t) = q(t)(H(t)t)^2$ where $q(t)$ is the deceleration parameter and $H(t)$ is the Hubble parameter. It is proved that the Friedmann equations (2.14) and (2.15) have an asymptotical solution $u(t)$ that satisfies the generalized power law if and only if the integral limit

$$\Upsilon = \lim_{t\to\infty} t \int_t^{\infty} \frac{\mu(s)}{s^2}\,\mathrm{d}s \tag{2.16}$$

exists and $\Upsilon < 1/4$. It is proved that the values of the constant Υ completely determines the asymptotical behavior of all cosmological parameters $u(t), H(t), q(t), p(t)$, and $\rho(t)$. It appears that this approach covers all results on cosmological parameters for the Standard model of the universe, as presented for example in [103] or in [139]. The crucial role in this analysis is played by the linear functional related to (2.16)

$$\mathbf{M}(f) = \lim_{t\to\infty} t \int_t^{\infty} \frac{f(s)}{s^2}\,\mathrm{d}s,$$

$\mathbf{M}$ is defined on the class of real functions that satisfy (2.16) for some Υ. It is proved that $f \in \ker \mathbf{M}$ if and only if there are real functions φ and η such that

$$f(t) = t\varphi'(t) + \eta(t), \quad \varphi(t), \eta(t) \to 0 \text{ as } t \to \infty.$$

This representation of $f \in \ker \mathbf{M}$ yields the asymptotical representations of the mentioned cosmological parameters, even assuming that the Einstein's cosmological constant Λ is non-zero. Detailed proofs and physical interpretations of these results can be found in [125].

2.6 More precise information about asymptotic behavior

2.6.1 De Haan type solutions

To some extent, the results here can be understood as ramifications and refinements of observations related to Corollary 2.2.

We start with the statements established for (2.2) by Geluk in [45], see also Marić [105], which essentially concern a description of behavior of $\mathcal{SV}$ solutions. Under the conditions posed on p (in addition to $t^2p(t) \to 0$ as $t \to \infty$, we have a second order condition), the Hartman result [53, Chapter XI, Ex. 9.9b] can be refined. Note that — for other comparison purposes — the Hartman result is recalled below in (2.33).

Theorem 2.11 (Geluk [45])**.** *Assume $t^2p(t) \to 0$ as $t \to \infty$. Let y be an eventually positive decreasing solution of* (2.2)*. If $p \in \mathcal{RV}(-2)$, then $-y \in \Pi(-ty'(t))$. Moreover, if*

$\int_a^\infty sp(s)\,\mathrm{d}s = \infty$, *then*

$$y(t) = \exp\left\{-\int_a^t sp(s)(1+\varepsilon(s))\,\mathrm{d}s\right\}, \tag{2.17}$$

and if $\int_a^\infty sp(s)\,\mathrm{d}s < \infty$, *then*

$$y(t) = y(\infty)\exp\left\{\int_t^\infty sp(s)(1+\varepsilon(s))\,\mathrm{d}s\right\}, \tag{2.18}$$

where $\varepsilon(t) \to 0$ *as* $t \to \infty$.

Note that if $\int_a^\infty sp(s)\,\mathrm{d}s$ converges in Theorem 2.11, then solutions under consideration tend to a positive constant, whereas if it diverges the representation does not imply in general that

$$y(t) \sim y(a)\exp\left\{-\int_a^t sp(s)\,\mathrm{d}s\right\}$$

as $t \to \infty$, see e.g. (2.19). A half-linear extension of the above result and related observations are presented in Section 3.6, while in Section 2.7 we offer a "4-th order extension." In Section 5.5 we discuss this type of formulas in connection with the so called nearly linear equations.

The following result is an extension of the previous theorem to the case of $\mathcal{RV}$ solutions.

Theorem 2.12 (Geluk [45])**.** *Assume* $L(t) := t^2p(t) - A \to 0$ *as* $t \to \infty$, *where* $A > 0$ *is a constant. Let* y *be an eventually positive decreasing solution of* (2.2). *If* $L \in \mathcal{SV}$, *then* $-t^\vartheta y(t) \in \Pi(-t(t^\vartheta y(t))')$ *and*

$$y(t) = t^{-\vartheta}\exp\left\{-\frac{1}{2\vartheta+1}\int_a^t \frac{L(s)}{s}(1+\varepsilon(s))\,\mathrm{d}s\right\},$$

where $\vartheta > 0$ *and* A *are related by* $\vartheta(\vartheta+1) = A$, *and* $\varepsilon(t) \to 0$ *as* $t \to \infty$.

A remark analogous to that after Theorem 2.11 holds also here. Further, see the remark after Theorem 2.17 which is concerned with certain generalization of the above theorem.

The class $\Pi\mathrm{R}_2$ opens further possibilities in obtaining more precise information about considered solutions y of (2.2). The following statement is a refinement of Theorem 2.11.

Theorem 2.13 (Geluk [45])**.** *Assume* $t^2p(t) \to 0$ *as* $t \to \infty$. *Let* y *be an eventually positive decreasing solution of* (2.2). *If* $-t^2p(t) \in \Pi(h)$, *then* $-y \in \Pi\mathrm{R}_2(v,w)$ *with* $v(t) = -ty'(t)$ *and*

$$w(t) \sim ty'(t) + t^2y''(t) \sim (h(t) + t^4p^2(t))y(t)$$

as $t \to \infty$. The following three cases are possible:
(i) If $\int_a^\infty s^3 p^2(s)\,\mathrm{d}s = \infty$, then

$$y(t) = \exp\left\{-\int_a^t sp(s)\,\mathrm{d}s + \int_a^t (s^3p^2(s) + h(s)/s)(1+\varepsilon(s))\,\mathrm{d}s\right\}, \tag{2.19}$$

where $\varepsilon(t) \to 0$ as $t \to \infty$.
(ii) If $\int_a^\infty s^3 p^2(s)\,\mathrm{d}s < \infty$ and $\int_a^\infty sp(s)\,\mathrm{d}s = \infty$, then

$$y(t) = \exp\left\{-\int_a^t sp(s)\,\mathrm{d}s + C - \int_t^\infty (s^3p^2(s) + h(s)/s)(1+\varepsilon(s))\,\mathrm{d}s\right\}, \tag{2.20}$$

where $C \in \mathbb{R}$ is a constant and $\varepsilon(t) \to 0$ as $t \to \infty$.
(iii) If $\int_a^\infty sp(s)\,\mathrm{d}s < \infty$, then

$$y(t) = y(\infty)\exp\left\{\int_t^\infty sp(s)\,\mathrm{d}s - \int_t^\infty (s^3p^2(s) + h(s)/s)(1+\varepsilon(s))\,\mathrm{d}s\right\} \tag{2.21}$$

where $\varepsilon(t) \to 0$ as $t \to \infty$.

Also in this setting we have an extension to the case of regularly varying solutions.

Theorem 2.14 (Geluk [45]). *Assume $L(t) := t^2p(t) - A \to 0$ as $t \to \infty$, where $A > 0$ is a constant. Let y be an eventually positive decreasing solution of* (2.2). *If $L \in \Pi(h)$, then $-t^\vartheta y(t) \in \Pi R_2$ and*

$$y(t) = t^{-\vartheta}\exp\left\{-\frac{1}{2\vartheta+1}\int_a^t \frac{L(s)}{s}\,\mathrm{d}s \right.$$
$$\left. + \frac{1}{(2\vartheta+1)^2}\int_a^t \frac{h(s) + (2\vartheta+1)^{-1}L^2(s)}{s}(1+\varepsilon(s))\,\mathrm{d}s\right\},$$

where $\vartheta > 0$ and A are related by $\vartheta(\vartheta+1) = A$ and $\varepsilon(t) \to 0$ as $t \to \infty$.

Under the conditions of Theorem 2.11, the linearly independent solution $x(t) = y(t)\int_a^t \frac{1}{y^2(s)}\,\mathrm{d}s$ satisfies $x(t)/t \in \Pi$. A representation can also be given. For example, under the additional condition $\int_a^\infty sp(s)\,\mathrm{d}s = \infty$,

$$x(t) = t(1 - 2t^2p(t)(1+o(1)))\exp\left\{\int_a^t (1+o(1))sp(s)\,\mathrm{d}s\right\}.$$

A similar representation can be given in other cases.

Assuming $p \in \mathcal{RV}(-2)$ and $t^2p(t) \to 0$ as $t \to \infty$ it follows from one of (2.17) and (2.18) that $-y \in \Pi(-ty'(t))$. Similarly $-t^2p(t) \in \Pi(h)$, $t^2p(t) \to 0$ as $t \to \infty$, and one of (2.19), (2.20), (2.21) imply $-y \in \Pi R_2(v, w)$. In the first case of Theorem 2.11 relation (2.17) implies

$$y(t) = y(a) \exp\left\{-(1 + o(1)) \int_a^t sp(s)\,\mathrm{d}s\right\}$$

but the last representation does not imply $-y \in \Pi$ any more. A similar remark applies to the other expansions. In Theorem 2.13 it follows from [11, Theorem 3] that $t^2 p(t) \sim \int_t^\infty h(s)/s\,\mathrm{d}s$ as $t \to \infty$, hence in (2.20) and in (2.21)

$$\int_t^\infty \frac{h(s)(1 + \varepsilon(s))}{s}\,\mathrm{d}s = (1 + o(1))t^2 p(t)$$

as $t \to \infty$.

We finish this section with refinements of the results for $\mathcal{RPV}(\infty)$ solutions of (2.2) using the classes $\mathcal{BSV}$ and Γ. The core of the following result is due by Omey and appeared in [47, 130, 131, 133]. Later it was extended to the half-linear case and additional observations were made (some of the being new also in the linear case), see Section 3.6. Denote

$$\mathbb{M}^+ = \{y : y \text{ is a solution of (2.2)}, y(t) > 0, y'(t) > 0 \text{ for large } t\}$$

and

$$\mathbb{M}^+_\infty = \{y \in \mathbb{M}^+ : \lim_{t\to\infty} y(t) = \infty\}.$$

Theorem 2.15 (Omey [47, 130, 131, 133])**.** *If $1/\sqrt{p} \in \mathcal{BSV}$, then $\emptyset \neq \mathbb{M}^+ = \mathbb{M}^+_\infty \subseteq \Gamma(1/\sqrt{p})$.*

A half-linear extension of the above theorem is presented in Subsection 3.6.1

Note that, under the assumptions of the theorem, for a solution $y \in \mathbb{M}^+$ we have

$$\lim_{t\to\infty} \frac{y'(t)}{\sqrt{p(t)}y(t)} = 1.$$

If $p \in C^1$, then the assumption from the previous theorem — in view of the properties of $\mathcal{BSV}$ functions — yields $(1/\sqrt{p(t)})' = -p'(t)p^{-3/2}(t)/2 \to A$, where $A = 0$, as $t \to \infty$, cf. Hartman, Wintner [54]. If $A > 0$, then we obtain regular variation of the solution; see Omey [130]. See also the discussion around equation (2.39).

Let y be a solution as in the previous theorem and consider the linearly independent solution

$$x(t) = C \int_t^\infty \frac{1}{y^2(s)}\,\mathrm{d}s,$$

$C > 0$. Note that $\lim_{t\to\infty} x'(t) = 0$ and x is nonprincipal solution while y is principal solution. From (1.25) we get $x \in \Gamma_-(1/\sqrt{p})$. Moreover, $x'(t)/(\sqrt{p(t)}x(t)) \to -1$ as $t \to \infty$.

In the following considerations (Omey [130, 133]) we get a second-order result by taking a closer look at $H(t) := y(t)/h(t) - y'(t)$, where $h \geq 0$ is defined by $h^2(t)p(t) = 1$. In order to obtain a rate of convergence result we assume that $h'(t) < 0$ for large t and that

$$-h' \in \mathcal{SC}(h).$$

Introduce the functions u and G by

$$u(t) = \exp\left\{\int_a^t \frac{1}{h(s)}\,\mathrm{d}s\right\} \quad \text{and} \quad G(t) = u(t)H(t).$$

Since $h'(t) \to 0$, we have $h \in \mathcal{SN}$ and $u \in \Gamma(h)$. Also we have

$$G'(t) = -\frac{u(t)h'(t)}{h^2(t)}y(t).$$

Note that $G'(t) > 0$ for large t and $G'(t) = u(t)(\sqrt{p})'y(t)$. [If $(\sqrt{p})'' > 0$, we have that $G''(t) > 0$ and G' is nondecreasing.] It is not difficult to see that

$$\frac{G'(t + \lambda h(t))}{G'(t)} \to \exp(2\lambda)$$

so that $G' \in \Gamma(h/2)$. Now this implies that $G(t) \sim G'(t)h(t)/2$. We conclude that

$$G(t) \sim -\frac{u(t)h'(t)y(t)}{2h(t)}$$

and then also that $H(t) \sim -h'(t)y(t)/(2h(t))$. It follows that

$$\frac{h(t)y'(t)}{y(t)} - 1 \sim \frac{h'(t)}{2},$$

or

$$\frac{y'(t)}{y(t)} = \frac{1}{h(t)} + (1 + \varepsilon(t))\frac{h'(t)}{2h(t)},$$

where $\varepsilon(t) \to 0$. Integration between t and $\lambda h(t)$ gives

$$\ln \frac{y(t + \lambda h(t))}{y(t)} - \lambda = I_1(t) + I_2(t),$$

where

$$I_1(t) = \int_0^\lambda \left(\frac{h(t)}{h(t + sh(t))} - 1\right)\mathrm{d}s,$$

$$I_2(t) = \int_0^t (1 + \varepsilon(t + sh(t)))\frac{h'(t + sh(t))h(t)}{h(t + sh(t))}\,\mathrm{d}s.$$

We get (see (1.28))

$$I_1(t) \sim -\frac{h'(t)\lambda^2}{2}, \quad I_2(t) \sim h'(t)\lambda.$$

It follows that

$$\ln\left(\frac{y(t+\lambda h(t))}{y(t)}\exp(-\lambda)\right) \sim h'(t)\left(\lambda - \frac{\lambda^2}{2}\right).$$

Since $h'(t) \to 0$, we obtain

$$\frac{y(t+\lambda h(t))}{y(t)}\exp(-\lambda) - 1 \sim h'(t)\left(\lambda - \frac{\lambda^2}{2}\right).$$

2.6.2 A different approach

Similarly as in the first part of the previous section also here we obtain more precise information about (slowly and regularly varying) solutions of (2.1). While previously the considerations were based on Π-variation here we offer an alterative procedure. Besides one method of successive approximations used in the proof of an auxiliary proposition, the properties of regularly varying solutions are the only essential tool.

To determine asymptotic behavior of solutions under consideration of equation (2.1), due to representations in Theorem 2.2 and Theorem 2.3, since $L_2(t) \sim 1/((1-\vartheta_1)L_1(t))$, one has to do it for L_1 only. In principle the same procedure applies for both slowly and regularly varying solutions. However, we consider separately two cases: $A = 0$ and $A \neq 0$, $-\infty < A < 1/4$ (see (2.7)). The reason is that in the former case one obtains a more general result than in the latter one.

We begin with an important auxiliary result which utilizes some of the ideas of Proposition 2.1 on p. 42.

Proposition 2.2 (Howard, Marić, Radaśin, [57, 105])**.** *Let the functions P and h be defined as Proposition 2.1. If there exists a continuous decreasing function $q(t)$ such that*

$$\int_t^\infty h^2(s)\,\mathrm{d}s \le q(t)h(t) \tag{2.22}$$

and

$$0 < q(t) \le C < 1/4, \tag{2.23}$$

and if for some $n \in \mathbb{N}$

$$\int^\infty q^n(t)h(t)\,\mathrm{d}t < \infty, \tag{2.24}$$

then the solution y of (2.1) *given by* (2.4) *satisfies*

$$y(t) \sim B\exp\left\{\int_a^t (P(s) - Z_{n-1}(s))\,\mathrm{d}s\right\}$$

as $t \to \infty$, where $Z_0(t) := 0$, $Z_n(t) := -\int_t^\infty (P(s)-Z_{n-1}(s))^2\,\mathrm{d}s$, $n \in \mathbb{N}$, $P(t) := \int_t^\infty p(s)\,\mathrm{d}s$, B is some positive constant, but one may take it to be 1.

Note that in general solutions y in the previous proposition need not to be slowly varying. But as established in Theorem 2.2, these are such under the condition (2.6). This, in addition, implies the existence of a function $q(t)$ satisfying (2.22) and (2.23) by taking $h(t) = q(t)/t$. Consequently, we get the following statement.

Theorem 2.16 (Howard, Marić, Radaśin, [57, 105]). *Let* (2.6) *be fulfilled. If condition for some $n \in \mathbb{N}$* (2.24)*, being here of the form $\int^\infty c^{n+1}(t)/t \,\mathrm{d}t < \infty$ holds, then two linearly independent solutions of* (2.1) *$y_1(t) = L_1(t)$, $y_2(t) = tL_2(t)$, where $L_1, L_2 \in \mathcal{NSV}$ with $L_2(t) \sim 1/L_1(t)$, possess the following asymptotic representation for $t \to \infty$*

$$y_1(t) \sim \exp\left\{\int_a^t (P(s) - Z_{n-1}(s))\,\mathrm{d}s\right\},$$
$$y_2(t) \sim t\exp\left\{-\int_a^t (P(s) - Z_{n-1}(s))\,\mathrm{d}s\right\}, \tag{2.25}$$

and $ty_1'(t)/y_1(t) \to 0$, $ty_2'(t)/y_2(t) \to 1$.

By Theorem 2.3, condition (2.7) which can be written in the form

$$\phi(t) := t\int_t^\infty p(s)\,\mathrm{d}s - A \to 0 \quad \text{as } t \to \infty,\ A \neq 0,$$

implies the existence of two linearly independent $\mathcal{NRV}$ solutions $y_i(t) = t^{\vartheta_i}L_i(t)$, $i = 1, 2$, $\vartheta_1 < \vartheta_2$ being the roots of $\vartheta^2 - \vartheta + A = 0$. Next we describe behavior of these solutions in a more precise way. The result was firstly proved in Geluk, Marić, Tomić [48] for the case $p(t) < 0$. The general case was treated in Marić [105]. Denote

$$\varrho(t) = \exp\left\{\int_1^t \frac{2(\vartheta_1 + \phi(s))}{s}\,\mathrm{d}s\right\}.$$

Theorem 2.17 (Marić [105]). *Let $A \in (-\infty, 1/4)$, $A \neq 0$. If*

$$\int^\infty \frac{\phi^2(s)}{s}\,\mathrm{d}s < \infty, \tag{2.26}$$

then (2.1) *possesses linearly independent solutions y_1, y_2 satisfying*

$$y_1(t) \sim t^{\vartheta_1}\exp\left\{\int_a^t \left(\frac{\phi(s)}{s} + 2\vartheta_1\int_s^\infty \frac{\varrho(\tau)}{\varrho(s)}\cdot\frac{\phi(\tau)}{\tau^2}\,\mathrm{d}\tau\right)\mathrm{d}s\right\}, \tag{2.27}$$

$$y_2(t) \sim \frac{t^{\vartheta_2}}{(1 - 2\vartheta_1)L_1(t)}$$

and $y_i'(t) \sim \vartheta_i y_i(t)/t$, $i = 1, 2$, as $t \to \infty$.

In Geluk, Marić, Tomić [48, Theorem 1.2] the condition of integrability (2.26) is replaced $\phi \in \mathcal{SV}$. That result can also be seen a a generalization of Theorem 2.11 and Theorem 2.12.

Note that in the asymptotic representation formula (2.27) two summands occurring in the exponential function need not in general to be of the same order or growth and consequently none of these can be disposed of. This is exemplified by the result which follows where also by strengthening the conditions, the behavior of solutions becomes much more legible.

Theorem 2.18 (Marić [105])**.** *If in addition to* (2.26) *one has*

$$\int_a^\infty \frac{\phi(t)}{\varrho(t)} \int_t^\infty \frac{\varrho(\tau)|\phi(\tau)|}{\tau^2}\,\mathrm{d}\tau\,\mathrm{d}t < \infty, \tag{2.28}$$

then

$$y_1(t) \sim t^\vartheta \exp\left\{\frac{1}{1-2\vartheta_1}\int_a^t \frac{\phi(s)}{s}\,\mathrm{d}s\right\} \tag{2.29}$$

as $t \to \infty$.

If $\phi(t)$ tends to zero sufficiently fast, e.g. like some power $t^{-\delta}$, $\delta > 0$ (or faster), then $\int^t \phi(s)/s\,\mathrm{d}s$ converges and (2.27) becomes $y_1(t) \sim Bt^{\vartheta_1}$. From that point of view, the following simple (and easy to apply) observation which uses properties of $\mathcal{RV}$ functions is of interest. If $\phi(t) = O(\Phi_0(t))$ as $t \to \infty$, where $\phi_0 \in \mathcal{SV}$ satisfies (2.26), then the behavior determined by (2.29) holds.

In the special case of $p(t) < 0$, the root ϑ_1 is always negative. Therefore, by the use of inequality $\varrho(t)/\varrho(s) \le (t/s)^{2\vartheta_1+2\varepsilon}$, which holds for any $\varepsilon > 0$ and $t > s, s \ge s_0(\varepsilon)$, condition (2.28) can be replaced by the simpler one

$$\int_a^\infty |\phi(t)| \int_t^\infty \frac{|\phi(s)|}{s^2}\,\mathrm{d}s\,\mathrm{d}t < \infty.$$

Theorem 2.18 (simplified by this remark) then extends the following result of Mařík and Ráb [117]. If for some $c > 0$

$$\int_a^\infty t|p(t) - c/t^2|\,\mathrm{d}t < \infty,$$

then (2.2) has a pair of solutions y_1, y_2 such that

$$y_i(t) \sim t^{-\vartheta_i} \text{ and } y'(t) \sim \vartheta_i t^{\vartheta_i - 1}, \tag{2.30}$$

where ϑ_i are the roots of $\vartheta^2 - \vartheta - c = 0$. Observe that if we put $\psi(t) = t(p(t) - ct^{-2})$, then the preceding condition implies

$$\int_a^\infty \int_s^\infty \frac{|\psi(\tau)|}{\tau}\,\mathrm{d}\tau\,\mathrm{d}s < \infty \text{ and } \int_t^\infty |\psi(s)|\,\mathrm{d}s \to 0$$

as $t \to \infty$. Consequently for $t \to \infty$,

$$|\phi(t)| = \left| t \int_t^\infty p(s)\,\mathrm{d}s - c \right| \le t \int_t^\infty \frac{|\psi(s)|}{s}\,\mathrm{d}s \le \int_t^\infty |\psi(s)|\,\mathrm{d}s \to 0$$

and also $\phi(t)/t = \int_t^\infty \phi(s)/s\,\mathrm{d}s$ is absolutely integrable over (a, ∞). Therefore all conditions of Theorem 2.18 are fulfilled and the result follows. This illustrates again kind of generalization we have: The slowly varying function $L(t) = \exp\left\{\frac{1}{1-2\vartheta_1}\int_a^t \phi(s)/s\,\mathrm{d}s\right\}$ in Theorem 2.18 multiplying by t^{ϑ_1} is in the result by Mařík and Ráb the special one which tends to a constant as $t \to \infty$.

In the rest of this section we assume $p(t) < 0$ and again we study slowly varying solutions. Of course, in this special case asymptotic representation of solutions remains the same as before (i.e., (2.25)). However here if y_1 is a $\mathcal{SV}$ solution we know according to the results of Section 2.1 that it is normalized and decreasing. Further, by the Representation Theorem, it is of the form $y_1(t) = \exp\left\{-\int_a^t \eta(s)/s\,\mathrm{d}s\right\}$, where $\eta(t)$ is positive and tends to zero as $t \to \infty$. It also satisfies the integral (Riccati type) equation

$$\eta(t) = t \int_t^\infty p(s)\,\mathrm{d}s - t \int_t^\infty \left(\frac{\eta(s)}{s}\right)^2 \mathrm{d}s.$$

Furthermore by Theorem 2.1, condition (2.6) is fulfilled. Therefore by putting $Z(t) - P(t) = \eta(t)/t$ and $Z_n(t) - P(t) = \eta_n(t)/t$, $n \in \mathbb{N}$, where Z is a solution of (2.5) and $P(t) = \int_t^\infty p(s)\,\mathrm{d}s$, solution y_1 has the form (2.4) and if a condition of the type (2.24) holds, the asymptotic representation for y_1 will follow. Since condition (2.6) is necessary and sufficient for the existence of $\mathcal{SV}$ solutions, one can reverse the above argument: Assume first (2.6) holds and then continue as above. It is worthwhile mentioning that not only for the existence but also for the asymptotic representation of $\mathcal{SV}$ solutions, condition (2.6) is the sole one needed. Properties of $\mathcal{SV}$ solutions do the rest. Condition (2.24) is only a technical one. We emphasize here also that in contrast to the general case (no sign condition on p) where all conditions that appear, (2.24) in particular, refer to functions h and q which have to be constructed, whereas here those conditions can be expressed in terms of the known function $P(t)$ as defined by (2.3).

Observe that condition (2.6) and the negativity of p imply that for any $\varepsilon > 0$ there exists t_0 such that for $t \ge t_0$, one has

$$\int_t^\infty P^2(s)\,\mathrm{d}s \le \varepsilon |P(t)|. \tag{2.31}$$

For, by a partial integration

$$\int_t^\infty P^2(s)\,\mathrm{d}s = -tP^2(t) + 2\int_t^\infty s(-P(s))(-p(s))\,\mathrm{d}s$$

and (2.31) follows.

Put

$$U_1(t) := \int_t^\infty P^2(s)\,\mathrm{d}s, \quad U_{n+1}(t) := 2\int_t^\infty -P(s)U_n(s)\,\mathrm{d}s,\ n \in \mathbb{N}.$$

Theorem 2.19 (Marić, Tomić [105, 115])**.** *If for some $n \in \mathbb{N}$*

$$\int^\infty U_n(s)\,\mathrm{d}s < \infty, \tag{2.32}$$

then for any $\mathcal{SV}$ solution y_1 of (2.1) *with $p(t) < 0$ and for the linearly independent solution the asymptotic representation* (2.25) *holds.*

Condition (2.32) might be cumbersome to verify. It can be replaced by a simpler in general a cruder one, as it is done in corollary below.

Estimate (2.31) implies the existence of a positive continuous function $q(t)$ decreasing to zero and such that $U_1(t) = \int_t^\infty P^2(s)\,\mathrm{d}s \le q(t)|P(t)|/2$. Then inequality $U_{n+1}(t) \le \varepsilon U_n(t)$ with $q(t)$ replacing ε, leads to $U_n(t) \le q^n(t)|P(t)|$. Hence the previous theorem implies the following statement.

Corollary 2.4. *If for some $n \in \mathbb{N}$, $\int^\infty q^n(t)|P(t)|\,\mathrm{d}t < \infty$, the asymptotic representation* (2.25) *holds.*

For comparison purposes we now recall the result by Hartman and Wintner ([53, Chapter XI, Ex. 9.9-b]); note that we utilized it already earlier, see the text before Theorem 2.11. Let $p(t)$ be a continuous complex function defined for $t \ge a$. If for some $\alpha \in [1,2]$ $\int^\infty t^{2\alpha-1}|p^\alpha(t)|\,\mathrm{d}t < \infty$, then equation (2.1) has a pair of solutions such that

$$\begin{aligned} y_1(t) &\sim \exp\left\{\int_a^t sp(s)\,\mathrm{d}s\right\} \ \text{ and } \ ty_1'(t)/y_1(t) \to 0, \\ y_2(t) &\sim t\exp\left\{-\int_a^t sp(s)\,\mathrm{d}s\right\} \ \text{ and } \ ty_2'(t)/y_2(t) \to 1 \end{aligned} \tag{2.33}$$

as $t \to \infty$.

The results for the derivatives show that when y_1, y_2 are real, they respectively are $\mathcal{NSV}$ and $\mathcal{NRV}(1)$.

E.g. for $p(t) < 0$ Theorem 2.19 gives the above behavior for y_1 and y_2 with $n = 1$. For, $y_1(t) \sim \exp\left\{\int_a^t p(s)\,\mathrm{d}s\right\}$. By integrating partially and then using (2.6) one obtain the above behavior for y_1, similarly for y_2 and for derivatives.

Instead of Hartman-Wintner conditions we have by (2.32) for $n = 1$, again after a partial integration, $\int_a^t sP^2(s)\,\mathrm{d}s < \infty$. These two conditions are not comparable in general. However, for the rather general example $p(t) = \varepsilon(t)/t^2$ where $\varepsilon(t)$ is almost decreasing, their condition is reduced to $\int^\infty \varepsilon^\alpha(s)/s\,\mathrm{d}s$ and ours to $\int^\infty \varepsilon^2(s)/s\,\mathrm{d}s < \infty$. They coincide for $\alpha = 2$ whereas for the remaining values of α it might happen that the latter is fulfilled but the former is not.

The asymptotics of solutions for $t \to \infty$ of the type we consider here is obtained for the more general system $y' = [A + B(t)]y$, where A is a constant $n \times n$ matrix whose characteristic roots are all simple and the continuous matrix $B(t)$ is such that $\int^{\infty} |B(s)| \, \mathrm{d}s < \infty$ and hence for the n-th order equation

$$y^{(n)} + b_1(t)y^{(n-1)} + \cdots + b_n(t)y = 0.$$

Compare with Coppel [23, Chapter IV, Th. 2, Th. 3]. We formulate explicitly the result for the n-th order equation with $n = 2$, $b_1(t) = 0$, $b_2(t) = p(t)$, in order to compare it with the present ones, e.g. that of Theorem 2.19 when $p(t) < 0$: If the function $p(t)$ is continuous for $t \geq t_0$ and $-\int^{\infty} sp(s) \, \mathrm{d}s < \infty$, then equation (2.1) has two linearly independent solutions y_1, y_2 such that

$$y_1(t) \to 1, \;\; y_2(t) \sim t, \;\; ty_1'(t) \to 0, \;\; y_2'(t) \to 1$$

as $t \to \infty$. On the other hand, Theorem 2.19, with $n = 1$, gives, as mentioned above: If $\int^{\infty} sP^2(s) \, \mathrm{d}s < \infty$, then for the above mentioned solutions y_1, y_2 one obtains (2.33). But in (the special case of) the Coppel theorem the integral $\int^{\infty} sp(s) \, \mathrm{d}s$ converges, this is to say that the $\mathcal{SV}$ functions $\exp\left\{\pm \int_a^t sp(s) \, \mathrm{d}s\right\}$ become the special ones which tend to a constant at variance with the present result where these can be completely general ones.

2.7 A note about higher order equations; an alternative approach to second order equations

At the end of the previous section we mentioned the concept of a higher order equation. Here we consider a general n-th order linear differential equation again, and briefly describe the approach by which it was investigated in Řehák [148]. The main ideas are simple: To apply the classical Poincaré-Perron type result [138, 140], to find a fundamental set of real solutions, and to use a suitable transformation. We will see that consequences of the main statement can yield known results (in particular for second order equations); the method however is different.

For a given function τ, we define the operator $\mathfrak{D}_\tau$ as $\mathfrak{D}_\tau u(s) = \tau(s)\frac{\mathrm{d}u}{\mathrm{d}s}$. Further, we set $\mathfrak{D}_\tau^n u(s) = \tau(s)\frac{\mathrm{d}}{\mathrm{d}s}\mathfrak{D}_\tau^{n-1} u(s)$, $n \in \mathbb{N}$, with $\mathfrak{D}_\tau^0 = \mathrm{id}$. Consider the linear equation

$$\mathfrak{D}_\tau^n u + \tilde{a}_{n-1}(s)\mathfrak{D}_\tau^{n-1} u + \cdots + \tilde{a}_1(s)\mathfrak{D}_\tau u + \tilde{a}_0(s)u = 0, \tag{2.34}$$

where $\tilde{a}_0, \ldots, \tilde{a}_{n-1}$ are continuous functions and τ is a positive continuous function on $[a, \infty)$.

Theorem 2.20 (Řehák [148]). *Let* $\lim_{t\to\infty} \tilde{a}_i(s) = A_i$, $i = 0, \ldots, n-1$, *and* ϱ_i *be the roots of* $\varrho^n + A_{n-1}\varrho^{n-1} + \cdots + A_1\varrho + A_0 = 0$ *all assumed to be real and of distinct moduli. Let*

$$\int_a^{\infty} \frac{1}{\tau(z)} \, \mathrm{d}z = \infty.$$

Then equation (2.34) *possesses a fundamental set of (real) solutions* $\{u_1, \dots, u_n\}$ *such that*

$$u_i \in \mathcal{NRV}_\omega(\varrho_i), \quad i = 1, \dots, n,$$

where $\omega(s) = \exp\left\{\int_a^s \frac{1}{\tau(z)}\,\mathrm{d}z\right\}$. *Every nontrivial (real) solution* u *of* (2.34) *satisfies* $|u| \in \mathcal{NRV}_\omega(\varrho_m)$ *for some* $m \in \{1, \dots, n\}$.

Note that one important step of the proof consists of transformation of equation (2.34) into

$$\frac{\mathrm{d}^n y}{\mathrm{d}t^n} + a_{n-1}(t)\frac{\mathrm{d}^{n-1} y}{\mathrm{d}t^{n-1}} + \cdots + a_1(t)\frac{\mathrm{d}y}{\mathrm{d}t} + a_0(s)y = 0 \tag{2.35}$$

by means of the relations $s = \xi(t)$ and $u(s) = y(\xi^{-1}(s)) = y(t)$, where ξ is defined as the inverse of the function $s \mapsto \int_a^s \frac{1}{\tau(z)}\,\mathrm{d}z$.

Theorem 2.20 can be applied in various ways. We mention several corollaries which are related to the results presented in this text.

Consider first the second order equation (such an equation is considered e.g. in (2.8))

$$u'' + b_1(s)u' + b_0(s)u = 0, \tag{2.36}$$

where b_0, b_1 are continuous on $[a, \infty)$.

Corollary 2.5. *Let there exist a function* τ *with the properties*

$$\tau \in C^1([a,\infty)),\ \tau(s) > 0 \text{ for } s \ge a, \text{ and } \int_a^\infty \frac{1}{\tau(z)}\,\mathrm{d}z = \infty$$

such that

$$\lim_{s\to\infty}(\tau(s)b_1(s) - \tau'(s)) = A_1 \neq 0,$$

$$\lim_{s\to\infty} \tau^2(s)b_0(s) = A_0,$$

where $A_1^2 > 4A_0$. *Then equation* (2.36) *possesses a fundamental set of solutions* $\{u_1, u_2\}$ *such that*

$$u_i \in \mathcal{NRV}_\omega(\varrho_i), \quad i = 1, 2,$$

where $\varrho_{1,2} = \frac{1}{2}\left(-A_1 \pm \sqrt{A_1^2 - 4A_0}\right)$ *and* $\omega(s) = \exp\left\{\int_a^s \frac{1}{\tau(z)}\,\mathrm{d}z\right\}$. *If* u *is a nontrivial (real) solution of* (2.36), *then* $|u|$ *is normalized regularly varying with respect to* ω *of index* ϱ_1 *or* ϱ_2.

If we choose $\tau(s) = s$, then $\omega(s) = s$ (up to a negligible multiplicative constant) and $\mathcal{NRV}_\omega = \mathcal{NRV}$. Hence we get the following corollary. Note that with this choice, the function $s = \xi(t)$ defining the new variable in the proof of Theorem 2.20 becomes $\xi(t) = e^t$. Recall also the two useful relations which hold generally: $\tau = \xi' \circ \xi^{-1}$ and $\omega = \exp \circ \xi^{-1}$.

Corollary 2.6. *If*

$$\lim_{s\to\infty} sb_1(s) = A_1 + 1 \neq 1, \quad \lim_{s\to\infty} s^2 b_0(s) = A_0 \tag{2.37}$$

with $A_1^2 > 4A_0$, then equation (2.36) *possesses a fundamental set of solutions $\{u_1, u_2\}$ such that $u_i \in \mathcal{NRV}(\varrho_i)$, $i = 1, 2$, where ϱ_1, ϱ_2 are as in the previous corollary. If u is a nontrivial (real) solution of* (2.36)*, then $|u|$ is normalized regularly varying of index ϱ_1 or ϱ_2.*

A more special choice in the previous corollary, namely $b_1(s) \equiv 0$, yields the existence of normalized $\mathcal{RV}$ solutions of the indices $\left(1 \pm \sqrt{1 - 4A_0}\right)/2$ to the equation $\frac{\mathrm{d}^2 u}{\mathrm{d}s^2} + b_0(s)u = 0$ provided $\lim_{s\to\infty} s^2 b_0(s) = A_0 \in (-\infty, 1/4)$. Recall that using a different technique the same result was obtained by Marić and Tomić [113] under the condition $b_0 < 0$, see also Corollary 2.2. This result was in essence first discovered by Omey in [130] (with additional conditions) and formulated for solutions tending to infinity. Later, the sufficient condition from [113] was improved and expressed in terms of $\lim_{s\to\infty} s \int_s^\infty b_0(z)\,\mathrm{d}z = A_0$, and this integral condition was shown to be also necessary for the existence of $\mathcal{RV}$ solutions, see Theorem 2.1.

Next we observe how Corollary 2.5 yields the result for a second order equation in the self-adjoint form

$$(r(s)u')' + p(s)u = 0, \tag{2.38}$$

where $p \in C, r \in C^1$, $r(s) > 0$ on $[a, \infty)$, when some special (and somehow optimal) setting is made. Recall that this equation was considered also in Section 2.3. Assume first $\int_a^\infty \frac{1}{r(z)}\,\mathrm{d}z = \infty$. Set $\omega(s) = \int_a^s \frac{1}{r(z)}\,\mathrm{d}z$. Because of the relation between ω and τ, which reads as $\tau = \frac{\omega}{\omega'}$, it means that $\tau(s) = r(s) \int_a^s \frac{1}{r(z)}\,\mathrm{d}z$. Equation (2.38) can be written as (2.36), where $b_1 = \frac{r'}{r}$ and $b_0 = \frac{p}{r}$. Since $\tau(s)b_1(s) = r'(s) \int_a^s \frac{1}{r(z)}\,\mathrm{d}z$, $\tau'(z) = 1 - r'(z) \int_a^s \frac{1}{r(z)}\,\mathrm{d}z$, and

$$\tau^2(s)b_0(s) = p(s)r(s) \left(\int_a^s \frac{1}{r(z)}\,\mathrm{d}z \right)^2,$$

we have that (4.49) reduces to $-1 = A_1$ and (4.50) reads as

$$\lim_{s\to\infty} p(s)r(s) \left(\int_a^s \frac{1}{r(z)}\,\mathrm{d}z \right)^2 = A_0.$$

Similarly, if $\int_a^\infty \frac{1}{r(z)}\,\mathrm{d}z < \infty$, then we set $\omega(s) = 1/\int_s^\infty \frac{1}{r(z)}\,\mathrm{d}z$, and (4.49) reduces to $1 = A_1$, while (4.50) becomes

$$\lim_{s\to\infty} p(s)r(s) \left(\int_s^\infty \frac{1}{r(z)}\,\mathrm{d}z \right)^2 = A_0.$$

Thus, applying Corollary 2.5, we get the following statement.

Corollary 2.7. *(i) Let* $\int_a^\infty \frac{1}{r(z)}\,\mathrm{d}z = \infty$*. Denote* $R(s) = \int_a^s \frac{1}{r(z)}\,\mathrm{d}z$*. If*

$$\lim_{s\to\infty} p(s)r(s)R^2(s) = A \in \left(-\infty, \frac{1}{4}\right),$$

then equation (2.38) *possesses a fundamental set of solutions* $\{u_1, u_2\}$ *such that* $u_i \in \mathcal{NRV}_R(\varrho_i)$*,* $i = 1, 2$*, where* $\varrho_{1,2} = \left(1 \pm \sqrt{1-4A}\right)$*. If* u *is a nontrivial (real) solution of* (2.38)*, then* $|u|$ *is normalized regularly varying with respect to* R *of index* ϱ_1 *or* ϱ_2*.*

(ii) Let $\int_a^\infty \frac{1}{r(z)}\,\mathrm{d}z < \infty$*. Denote* $\tilde{R}(s) = \int_s^\infty \frac{1}{r(z)}\,\mathrm{d}z$*. If*

$$\lim_{s\to\infty} p(s)r(s)\tilde{R}^2(s) = \tilde{A} \in \left(-\infty, \frac{1}{4}\right),$$

then equation (2.38) *possesses a fundamental set of solutions* $\{u_1, u_2\}$ *such that* $u_i \in \mathcal{NRV}_{1/\tilde{R}}(\tilde{\varrho}_i)$*,* $i = 1, 2$*, where* $\tilde{\varrho}_{1,2} = \left(-1 \pm \sqrt{1-4\tilde{A}}\right)$*. If* u *is a nontrivial (real) solution of* (2.38)*, then* $|u|$ *is normalized regularly varying with respect to* $1/\tilde{R}$ *of index* $\tilde{\varrho}_1$ *or* $\tilde{\varrho}_2$*.*

As we could see in Theorem 2.8, using a different approach (more precisely, a combination of the Riccati technique with the contraction mapping theorem), Jaroš and Kusano in [60], showed that the condition

$$\lim_{s\to\infty} R(s) \int_s^\infty p(z)\,\mathrm{d}z = A < \frac{1}{4}$$

is sufficient and necessary for the existence of a fundamental set of solutions with the properties from the last corollary, provided $\int_a^\infty \frac{1}{r(z)}\,\mathrm{d}z = \infty$. Since — provided the latter limit exists and $\int_a^\infty p(z)\,\mathrm{d}z$ converges — the L'Hospital's rule yields

$$\lim_{s\to\infty} R(s) \int_s^\infty p(s)\,\mathrm{d}z = \lim_{s\to\infty} p(s)r(s)R^2(s);$$

we see that we offer an alternative (and quite simple) approach to the result closely related to Theorem 2.8. Similarly, if we assume $\int_a^\infty \frac{1}{r(z)}\,\mathrm{d}z < \infty$, then the necessary and sufficient condition for the existence of generalized $\mathcal{RV}$ solutions from Theorem 2.8 reads as

$$\lim_{s\to\infty} (\tilde{R}(s))^{-1} \int_s^\infty \tilde{R}^2(z)p(z)\,\mathrm{d}z = \tilde{A} < \frac{1}{4}.$$

To see a relation with Corollary 2.7-(ii), note that

$$\lim_{s\to\infty} (\tilde{R}(s))^{-1} \int_s^\infty \tilde{R}^2(z)p(z)\,\mathrm{d}z = \lim_{s\to\infty} p(s)r(s)\tilde{R}^2(s),$$

provided the assumptions for the use of the L'Hospital rule are satisfied.

Corollary 2.5 can be utilized to study self-adjoint form (2.38) also in the following, slightly different, way. We set $\tau(s) = s$, which yields $\omega(s) = s$. Recall that equation (2.38) can be written as (2.36), where $b_1 = \frac{r'}{r}$ and $b_0 = \frac{p}{r}$.

Since then $\tau(s)b_1(s) - \tau'(s) = \frac{sr'(s)}{r(s)} - 1$ holds, condition (4.49) can be understood as $r \in \mathcal{NRV}(A_1 + 1)$. Condition (4.50) reads as $p(s) \sim A_0 s^{-2} r(s)$ which among others means that also p is required to be regularly varying, namely of index $(A_1 + 1) - 2 = A_1 - 1$. Thus we get the following statement.

Corollary 2.8. *If*

$$r \in \mathcal{NRV}(A_1 + 1) \text{ and } s^2 p(s) \sim A_0 r(s) \text{ as } s \to \infty,$$

with $A_1^2 > 4A_0$, then equation (2.38) *possesses a fundamental set of solutions $\{u_1, u_2\}$ such that $u_i \in \mathcal{NRV}(\varrho_i)$, $i = 1, 2$, where ϱ_1, ϱ_2 are as in Corollary 2.5.*

We conclude this part devoted to second order equations with showing another way how Corollary 2.5 can be utilized. In addition, we present related observations and indicate relations with known results. We consider the equation

$$u'' = p(s)u, \tag{2.39}$$

where $p > 0$, $p \in C^1$. If $b_1 = 0$ and $b_0 = -p$, then (2.36) reduces (2.39). Clearly, equation (2.39) is nonoscillatory. Set $\tau = 1/\sqrt{p}$. Then condition (4.49) reads as

$$\lim_{s\to\infty} \left(\frac{-1}{\sqrt{p(s)}} \right)' = A_1 \tag{2.40}$$

and condition (4.50) reduces to $A_0 = -1$. Assume first $A_1 \neq 0$. Then $\int_a^\infty \frac{\mathrm{d}z}{\tau(z)}\,\mathrm{d}z = \infty$ and (2.39) possesses solutions $u_i \in \mathcal{NRV}_\omega(\vartheta_i)$, $i = 1, 2$, where

$$\vartheta_{1,2} = \frac{1}{2}\left(-A_1 \pm \sqrt{A_1^2 + 4}\right)$$

(with $\vartheta_{1,2} \neq -1, 0, 1$) and $\omega(s) = \exp\left\{\int_a^s \sqrt{p(z)}\,\mathrm{d}z\right\}$, by Corollary 2.5. Hence, $\frac{u_i'(s)}{u_i(s)} \sim \vartheta_i \sqrt{p(s)}$ as $s \to \infty$, $i = 1, 2$. Since u_i is positive, u_i' is eventually positive or negative. We have

$$\frac{u_i''(s)u_i(s)}{(u_i'(s))^2} = \frac{p(s)u_i^2(s)}{(u_i'(s))^2} \sim \frac{1}{\vartheta_i^2} \neq 1, \tag{2.41}$$

$i = 1, 2$. We then get $u_i \in \mathcal{RV}\left(\frac{\vartheta_i^2}{\vartheta_i^2 - 1}\right)$, $i = 1, 2$, cf. Omey [130] where increasing solutions are considered. Because of monotonicity of u', we have

$$u_i \in \mathcal{NRV}\left(\frac{\vartheta_i^2}{\vartheta_i^2 - 1}\right), \tag{2.42}$$

$i = 1, 2$. Observe that condition (2.40) with $A_1 \neq 0$ implies the existence of the limit $\lim_{s\to\infty} s^2 p(s)$; compare with the latter condition in (2.37). The expected correspondence between the indices of regular variation in (2.42) and in Corollary 2.6 with $b_1 = 0$ and $b_0 < 0$ can be now easily revealed; the details are left to the reader.

Assume now $A_1 = 0$ in (2.40) and $\int_a^\infty \sqrt{p(z)}\,\mathrm{d}z = \infty$. Then by Hartman-Wintner [54, Paragraph 24], (2.39) has a pair of solutions satisfying

$$u_i'(s) \sim \pm\sqrt{p(s)}u_i(s) \tag{2.43}$$

as $s \to \infty$, $i = 1, 2$. Notice that the roots of the associated characteristic equation $\varrho^2 - 1 = 0$ are ± 1, and so Corollary 2.5 can not be used. In terms of generalized regular variation, we can write $u_i \in \mathcal{NRV}_\omega(\pm 1)$, where $\omega(s) = \exp\left\{\int_a^s \sqrt{p(z)}\,\mathrm{d}z\right\}$. For increasing solutions, this statement was rediscovered in [130] by Omey. In addition, by the same author, relations with the class Γ were shown and condition (2.40) with $A_1 = 0$ was relaxed to $1/\sqrt{p} \in \mathcal{BSV}$ (without requiring $p \in C^1$) in [131], see Theorem 2.15. Other type of relaxation of (2.40) with $A_1 = 0$ can be found in Hartman-Wintner [54], namely to the condition

$$\sup_{0 \le k < \infty} \frac{|\ln p(s+k)/p(s)|}{1 + \int_s^{s+k} \sqrt{p(z)}\,\mathrm{d}z} \to 0$$

as $s \to 0$, which is sufficient and also necessary for equation (2.39) to posses a pair of solutions satisfying (2.43). Let us show how the classes $\Gamma(\pm 1; v)$ are involved. In the set $\{u_1, u_2\}$ satisfying (2.43), clearly one solution must be increasing (say u_1) while other is decreasing. Similarly as in (2.41), we obtain $\frac{u_i''(s)u_i(s)}{(u_i'(s))^2} \sim 1$ as $s \to \infty$, $i = 1, 2$. Hence we get $u_i \in \Gamma(\sigma_i; v)$, $i = 1, 2$, where $\sigma_1 = 1, \sigma_2 = -1$, and $v = |u/u'|$. Thus

$$u_i = \Gamma(\sigma_i; p^{-\frac{1}{2}}),$$

$i = 1, 2$, by (2.43). From the transformation relations between $u(s)$ and $y(t)$ (see the text after (2.35)) we obtain a pair of Poincaré-Perron solutions $\{y_1, y_2\}$ of

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + a_1(t)\frac{\mathrm{d}y}{\mathrm{d}t} - y = 0 \tag{2.44}$$

satisfying $\frac{\mathrm{d}y(t)}{\mathrm{d}t} \cdot \frac{1}{y(t)} \to \pm 1$ as $t \to \infty$, where $a_1(t) = \tilde{a}_1(\xi(t)) = -\tau'(\xi(t)) \to 0$ as $t \to \infty$; ξ being the inverse of $s \mapsto \int_a^s \frac{1}{\tau(z)}\,\mathrm{d}z$. In this connection we recall the result [23, Chapter IV, Theorem 2] according to which (2.44) possesses a pair of solutions satisfying $y(t) \sim e^t$, $y_2(t) \sim e^{-t}$, $\frac{\mathrm{d}y_1(t)}{\mathrm{d}t} \sim e^t$, $\frac{\mathrm{d}y_2(t)}{\mathrm{d}t} \sim e^{-t}$ as $t \to \infty$ provided the (stronger) condition $\int_a^\infty |a_1(z)|\,\mathrm{d}z < \infty$ is fulfilled. Clearly then $y_1 \in \Gamma(1; 1)$, $y_2 \in \Gamma(-1; 1)$. Another result of Coppel ([23, Chapter IV, Theorem 14]) says that if $p > 0$ and $\int_a^\infty |p^{-3/2}(t)p''(t)|\,\mathrm{d}t < \infty$, then (2.39) has a fundamental set of solutions satisfying

$$u_i(s) \sim p^{-1/4}(s)\exp\left\{\pm\int_a^s p^{1/2}(t)\,\mathrm{d}t\right\},$$

$$u_i'(s) \sim p^{1/4}(s)\exp\left\{\pm\int_a^s p^{1/2}(t)\,\mathrm{d}t\right\}$$

as $s \to \infty$, $i = 1, 2$.

For completeness, we mention another two classical results which are somehow related to the topic of this paragraph. Hartman's result [53, Ex. 9.9-b] is recalled in (2.33). Mařík's and Ráb's statement reads as (2.30) The conditions in these statements (including Coppel's one) differ among each other but lead to the same type of conclusions. As we could see above, such a behavior is typical for regularly (or rapidly) varying solutions. Concerning the requirement imposed on p, the behavior of $s^2 p(s)$ (or, possibly, of $s \int_s^\infty p(t)\,\mathrm{d}t$ or of $s \int_s^{\lambda s} p(t)\,\mathrm{d}t$) is crucial in that respect. Recall that the existence of the finite limit $\lim_{s\to\infty} s^2 p(s)$ leads to a regularly varying solutions while $\lim_{s\to\infty} s^2 p(s) = \infty$ yields rapidly varying solutions of (2.39), see Section 2.1. The fact that a solution is a $\mathcal{RV}$ or $\mathcal{RPV}$ function is less precise than the formulas in the quoted results, but requires milder hypotheses.

Next we examine the third order equation in the self-adjoint form

$$u''' + 2b(s)u' + b'(s)u = 0, \tag{2.45}$$

where $b \in C^1([a, \infty))$. First note that this equation is not so special in comparison with the general equation

$$u''' + b_2(s)u'' + b_1(s)u' + b_0(s)u = 0, \tag{2.46}$$

as it might seem. Indeed, the second derivative term in (2.46) can always be removed by the transformation

$$u(s) = v(s) \exp\left\{-\frac{1}{3}\int_a^s b_2(z)\,\mathrm{d}z\right\}$$

to obtain the equation $v''' + 2b(s)v' + (b'(s) + c(s))v = 0$. This form has frequently occurred in the literature ([65, 157]), thus we can easily make a comparison. Our method (which leads to the existence of generalized regularly varying solutions) and is based on Theorem 2.20 applies also to general equation (2.46).

Corollary 2.9. *If*

$$\lim_{s\to\infty}(-s^3 b'(s)) = B < 1, \tag{2.47}$$

then (2.45) *possesses a fundamental system of solutions* $\{u_1, u_2, u_3\}$ *such that* $u_i \in \mathcal{NRV}(\varrho_i)$, $i = 1, 2, 3$, *where* $\varrho_1 = 1$, $\varrho_2 = 1 + \sqrt{1 - B}$, $\varrho_3 = 1 - \sqrt{1 - B}$. *If u is a nontrivial (real) solution of* (2.45), *then* $|u|$ *is normalized regularly varying of index* 1 *or* $1 + \sqrt{1 - B}$ *or* $1 - \sqrt{1 - B}$.

In Jaroš, Kusano, Marić [65], see Section 5.3 — using a different approach which is based on the correspondence of (2.45) with certain second order equation — it is proved that the statement of Corollary 2.9 holds under the assumption $\lim_{s\to\infty} \int_s^\infty b(z)\,\mathrm{d}z = \frac{B}{2}$. A simple use of the L'Hospital rule shows how (2.47) implies this assumption. Note however that (2.47) does not require the convergence of $\int_a^\infty b(z)\,\mathrm{d}z$.

We conclude this section with observations related to the two-term fourth order linear differential equation

$$u^{(4)} + b(s)u = 0, \tag{2.48}$$

where $b \in C([a,\infty))$. Denote

$$G(x) = x^4 - 6x^3 + 11x^2 - 6x.$$

Applying Theorem 2.20 we get the following statement.

Corollary 2.10. *If*

$$\lim_{s\to\infty} s^4 b(s) = A \in \left(-\frac{9}{16}, 1\right),$$

then equation (2.48) *possesses a fundamental set of solutions* $\{u_1, u_2, u_3, u_4\}$ *such that* $u_i \in \mathcal{NRV}(\varrho_i)$, $i = 1,2,3,4$, *where* ϱ_i *are the (real) roots of* $G(\varrho) + A = 0$. *If* u *is a nontrivial (real) solution of* (2.48), *then* $|u|$ *is normalized regularly varying of index* ϱ_1 *or* ϱ_2 *or* ϱ_3 *or* ϱ_4.

Note that as a by-product of Corollary 2.10 we get a nonoscillation criterion for equation (2.48). For related criteria see [157, Chapter 3].

Let us now examine slowly varying solutions of (2.48) where $b < 0$ and $A = 0$. Let us write the equation in the form

$$u^{(4)} = p(s)u, \tag{2.49}$$

i.e., $p = -b > 0$. Since $A = 0$, the roots of $G(\varrho) + A = 0$ are $0, 1, 2, 3$, and so $\mathcal{SV}$solutions of (2.49) indeed exist. Eventually positive solutions of (2.49) are eventually monotone and therefore a solution $u \in \mathcal{SV}$ is eventually monotone. It can be shown that $\mathcal{SV}$ solutions cannot increase. Hence, if we deal with $\mathcal{SV}$ solutions, in fact, we deal with all positive decreasing solutions of (2.49), since non-$\mathcal{SV}$ solutions are in $\mathcal{RV}(1) \cup \mathcal{RV}(2) \cup \mathcal{RV}(3)$. As for the existence of $\mathcal{SV}$ solutions, the condition $\lim_{s\to\infty} s^4 p(s) = 0$ can be relaxed to

$$\lim_{s\to\infty} \int_s^\infty z^2 p(z)\,\mathrm{d}z = 0. \tag{2.50}$$

More precisely, it then holds that any eventually positive decreasing solution u of (2.49) is in $\mathcal{NSV}$.

Next we give asymptotic formula for a solution $u \in \mathcal{SV}$ under the assumption $p \in \mathcal{RV}(-4)$. Denote $H(s) = s^3 p(s)/6$. It holds $H \in \mathcal{RV}(-1)$ and any of the possibilities, convergence or divergence of $\int_a^\infty H(z)\,\mathrm{d}z$, can in general occur.

Theorem 2.21 (Řehák [148])**.** *Let* $p \in \mathcal{RV}(-4)$ *with* $L_p(s) \to 0$ *as* $s \to \infty$; L_p *being the* $\mathcal{SV}$ *component of* p. *Then the set of eventually positive decreasing solutions of* (2.49) *(which is nonempty) is a subset of* $\mathcal{SV}$. *For each solution* $u \in \mathcal{SV}$ *(which is necessarily decreasing) one has* $-u \in \Pi(-su'(s))$ *and one of the following formula holds:*

(i) If $\int_a^\infty H(z)\,\mathrm{d}z = \infty$*, then* $u(s)$ *tends to zero as* $s \to \infty$ *and satisfies the formula*

$$u(s) = \exp\left\{-\int_a^s (1 + o(1))H(z)\,\mathrm{d}z\right\}.$$

(ii) If $\int_a^\infty H(z)\,\mathrm{d}z < \infty$*, then* $u(s)$ *tends to a positive constant* $u(\infty)$ *as* $s \to \infty$ *and satisfies the formula*

$$u(s) = u(\infty)\exp\left\{\int_s^\infty (1 + o(1))H(z)\,\mathrm{d}z\right\}.$$

The above theorem can be understood as a "fourth-order extension" of Theorem 2.11.

2.8 On zeros of oscillatory solutions, asymptotic behavior of maxima and of eigenvalues

The problem of determining the asymptotic behavior of the number of zeros is an old one and studied by many authors, see e.g. Hartman [53, Chapter XI]. Here we show how the class of Beurling slowly varying functions can be utilized to obtain some information about this behavior. The results were achieved by Hačik and Omey [51], see also Marić [105], Omey [130].

In this section we assume that p in (2.1) is a positive and continuous function.

In the sequel, $\{t_n\}$, $t_n \to \infty$, denotes the sequence of zeros of a solution $y(t)$ of (2.1) and $N(I)$ the number of these in the interval I.

As an example, consider equation (2.1) with $p(t) = k^2, k \in (0, \infty)$. Every solution of (2.1) has the form $y(t) = c_1 \sin(kt) + c_2 \cos(kt)$, where $c_c, c_2 \in \mathbb{R}$. We see that this solution has an infinite number of zeros t_n, $n \in \mathbb{N}$, which satisfy $t_{n+1} - t_n = \pi/k$ and $kt_n \sim n$ as $n \to \infty$. This last statement says that the number of zeros less than t_n asymptotically equals $\sqrt{p(t_n)}t_n$.

In the general case Titchmarsh [165], p. 146, proved that, whenever $p(t)$ is continuous and of bounded variation, there holds that

$$\left|N((a,t)) - \frac{1}{\pi}\int_a^t \sqrt{p(s)}\,\mathrm{d}s\right| \le 1 + \frac{1}{2\pi}\int_a^t 1/\sqrt{p(s)}\,\left|\mathrm{d}(\sqrt{p(s)})\right|.$$

We will discuss conditions under which we can relate the asymptotic behavior of $p(t)$ to that of $N((0,t])$, to that of the sequence of zeros $\{t_n\}$ and to that of the sequence $\{t_{n+1} - t_n\}$.

First note that if $1/\sqrt{p} \in \mathcal{BSV}$, then $t^2 p(t) \to \infty$ as $t \to \infty$ so that equation (2.1) is oscillatory.

Theorem 2.22 (Hačik and Omey [51])**.** *Suppose* $1/\sqrt{p} \in \mathcal{BSV}$*. Then*

(i) $N((0,t]) \sim \frac{1}{\pi}\int_0^t \sqrt{p(s)}\,\mathrm{d}s$ *as* $t \to \infty$*,*

(ii) $N((t, t + s/\sqrt{p(s)}]) = \frac{t}{\pi} + O(1)$ *for each* $s > 0$*.*

From the properties of $\mathcal{BSV}$ functions we have that if $1/\sqrt{p} \in \mathcal{BSV}$, then p may be regularly varying and likewise belong to the class Γ. This justifies the following two corollaries.

Corollary 2.11. *(i) If $p \in \mathcal{RV}(\alpha)$, $\alpha > -2$, then $N((0,t]) \in \mathcal{RV}((\alpha+2)/2)$ and*

$$N((0,t]) \sim \frac{2}{\pi(\alpha+2)} t \sqrt{p(t)}$$

as $t \to \infty$

(ii) If $p \in \Gamma(h(t))$, then $N((0,t]) \in \Gamma(2h(t))$ and

$$N((0,t]) \sim \frac{2}{\pi} h(t) \sqrt{p(t)}$$

as $t \to \infty$.

The proof of this corollary follows from the previous theorem and elementary properties of the classes $\mathcal{RV}$ and Γ.

If we replace t by t_n and use $\lim_{n\to\infty} \sqrt{p(t_n)}(t_{n+1} - t_n) = \pi$, we get the following statement for the sequence of zeros.

Corollary 2.12. *(i) If $p \in \mathcal{RV}(\alpha)$, $\alpha > -2$, then $t_n \sqrt{p(t_n)} \sim n\pi(\alpha+2)/2$ and*

$$n \frac{t_{n+1} - t_n}{t_n} \to \frac{2}{\alpha+2}$$

as $n \to \infty$.

(ii) If $p \in \Gamma(h(t))$, then $h(t_n) \sqrt{p(t_n)} \sim n\pi/2$ and

$$n \frac{t_{n+1} - t_n}{h(t_n)} \to 2$$

as $n \to \infty$.

Up to now we assumed $1/\sqrt{p(t)} \in \mathcal{BSV}$ which implies $t^2 p(t) \to \infty$ as $t \to \infty$. When $t^2 p(t) \to A \geq 1/4$ however (2.1) may remain oscillatory. To deal with this kind of equations we transform (2.1) into a more suitable form. Generally, consider the differential equation (2.9), $r(t) > 0$, $t \geq a \geq 0$. For a suitable function Ψ we can define

$$\xi(t) = \int_a^t \frac{1}{r(s)\Psi^2(s)} \, \mathrm{d}s, \quad \eta(\xi) = \frac{y(t)}{\Psi(t)}.$$

With this transformation, (2.9) becomes

$$\frac{\mathrm{d}^2\eta}{\mathrm{d}\xi^2} + \tilde{p}(\xi)\eta(\xi) = 0, \quad 0 \leq \xi \leq \xi(\infty),$$

where $\tilde{p}(\xi) = [(r\Psi')' + p\Psi]\Psi^3 r$, see Willett [166, p. 597]. Now t_n is a zero of y if and only if $\xi_n \equiv \xi(t_n)$ is a zero of η. Hence the number of zeros of y less than or equal T is the same as the number of zeros of η less than $\xi(T)$. Now Theorem 2.22 yields

Theorem 2.23 (Hačik and Omey [51]). *Suppose that $\tilde{p}$ satisfies the conditions of Theorem 2.22. Then for equation* (2.9) *we have*

$$N((0,t]) \sim \frac{1}{\pi}\int_a^t \sqrt{\tilde{p}(\xi(s))}\xi'(s)\,\mathrm{d}s$$

as $t \to \infty$.

In the case of (2.1) we have $r \equiv 1$. If we choose $\Psi(t) = \sqrt{t}$, $a = 1$, then $\xi = \ln t$ and $\tilde{p}(\xi) = t^2 p(t) - 1/4$. Hence we get

Corollary 2.13. *(i) If $\tilde{p}(t) = e^{2t}p(e^t) - 1/4 \in \mathcal{RV}(\alpha)$, $\alpha > -2$, then*

$$N((0,t]) \sim \frac{1}{\pi}\ln t\sqrt{t^2 p(t) - 1/4}\frac{2}{\alpha+2}$$

as $t \to \infty$.

(ii) If $\tilde{p} \in \Gamma(h)$, then

$$N((0,t]) \sim \frac{1}{\pi}2h(\ln t)\sqrt{t^2 p(t) - 1/4}$$

as $t \to \infty$.

This corollary of course remains valid when $t^2 p(t) \to \infty$.

Omey in [130] studies also asymptotic behavior of the sequence $\{y^2(s_n)\}$ of successive maxima of y^2, y being a solution of (2.1), s_n denotes the value at which $|y(t)|$ reaches a maximum. For illustration, we present here one selected result. Note that its proof uses some of the previous results in this section, the Sturm comparison theorem and Wiman's method [167].

Theorem 2.24 (Omey [130]). *(i) Let p be a monotone function such that*

$$\lim_{t\to\infty}\frac{tp'(t)}{p(t)} = \alpha \in (-2,\infty).$$

Then the sequences $\{y'^2(t_n)\}$, $\{y^{-2}(s_n)\}$ are in $\mathcal{RV}(\alpha/(\alpha+2))$.

(ii) Let $p' \in \Gamma(h(t))$ and

$$\lim_{t\to\infty}\frac{\ln(p(t))}{h(t)\sqrt{p(t)}} = 0.$$

Then the sequences $\{y'^2(t_n)\}$, $\{y^{-2}(s_n)\}$ are in $\mathcal{RV}(1)$.

Omey [130] further studies the asymptotic behavior of the eigenvalues of the linear operator

$$\mathcal{L} = q(t) - \frac{\mathrm{d}^2}{\mathrm{d}^2 t}$$

in $(0,\infty)$. Consider the differential equation $\mathcal{L}[y] = \lambda y$, where $\lambda \in \mathbb{R}$. A function which satisfies this equation and also some boundary conditions (e.g. $y(0) = 0$,

$y'(0) = 0$) is called an eigenfunction. The corresponding value of λ is called an eigenvalue. We will assume that $q(t)$ diverges to infinity as $t \to \infty$, since the operator $\mathcal{L}$ then has an infinite number of eigenvalues $\{\lambda_n\}$; also the eigenfunction associated with λ_n has exactly n zeros [165, p. 110].

In the theorem below it is examined how the distribution of the eigenvalues $\{\lambda_n\}$ is determined by the function q. We present the result without a proof; note just that it utilizes the Sturm comparison theorem, an Abelian type theorem involving $\mathcal{RV}$ functions, and some the previous results in this section. Let $M(x)$ denote the number of eigenvalues of the operator $\mathcal{L}$ not exceeding x.

Theorem 2.25 (Omey [130])**.** *Let q be a continuous, increasing function with $q(0) = 0$. If $q \in \mathcal{RV}(\alpha)$, $\alpha \in (0, \infty)$, then $M \in \mathcal{RV}((\alpha + 2)/(2\alpha))$. Furthermore,*

$$q(M(t)/\sqrt{t}) \sim t \left(\frac{B(3/2, 1/\alpha)}{\alpha\pi} \right)^{\alpha}$$

as $t \to \infty$, where $B(\cdot, \cdot)$ denotes the beta function.

Under the conditions of this theorem, the sequence $\{\lambda_n\}$ is regularly varying with the index $2\alpha/(\alpha + 2)$ and

$$q(n/\sqrt{\lambda_n}) \sim k_n \left(\frac{B(3/2, 1/\alpha)}{\alpha\pi} \right)^{\alpha}$$

as $n \to \infty$.

Solving this "asymptotic functional equation", we can derive an explicit asymptotic formula for $\{\lambda_n\}$; we give a sufficient condition to do this (the proof uses the concept of conjugate $\mathcal{SV}$ solutions). Suppose $q(t) = (tL(t))^{\alpha}$, $\alpha \in (0, \infty)$, where $L \in \mathcal{SV}$. If for all real β,

$$\lim_{t\to\infty} \frac{L(tL^{\beta}(t))}{L(t)} = 1,$$

then

$$\lambda_n \sim \left(\frac{n\alpha\pi}{B(3/2, 1/\alpha)} \right)^{2\left(\frac{\alpha}{\alpha+2}\right)^2} \left(q\left(\left(\frac{n\alpha\pi}{B(3/2, 1/\alpha)} \right)^{\frac{2}{\alpha+2}} \right) \right)^{\frac{2}{\alpha+2}}$$

as $n \to \infty$.

Chapter 3

Half-linear second order differential equations

3.1 Introduction

In this chapter we consider the half-linear second order differential equation

$$(r(t)\Phi(y'))' + p(t)\Phi(y) = 0, \quad \Phi(u) := |u|^{\alpha-1}\operatorname{sgn} u,\ \alpha > 1, \tag{3.1}$$

Recall that the terminology *half-linear* differential equation (systematically used by Bihari and Elbert for the first time) reflects the fact that the solution space of (3.1) is homogeneous, but not additive. The works of Mirzov [126] and Elbert [30] are usually regarded as pioneering ones. A quite comprehensive treatment on half-linear differential equations can be found in the book [28] by Došlý, Řehák.

We suppose that the functions r, p are continuous and $r(t) > 0$ in the interval under consideration. Many of the results for (3.1) can be formulated under weaker assumptions that the functions $1/r$, c are locally integrable. However, since we are interested in solutions of (3.1) in the classical sense (i.e., a solution y of (3.1) is a C^1 function such that $r\Phi(y') \in C^1$ and satisfies (3.1) in an interval under consideration), the continuity assumption is appropriate for this setting.

Half-linear equations are closely related to the partial differential equations with p-Laplacian. In fact, (3.1) is sometimes called the *differential equation with the one-dimensional p-Laplacian*. Recall that the α-Laplacian is a partial differential operator of the form

$$\Delta_\alpha u := \operatorname{div}\left(\|\nabla u\|^{\alpha-2}\nabla u\right),$$

where, for $u = u(x) = u(x_1, \ldots, x_N)$, $\nabla u = \left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_N}\right)$ is the Hamilton nabla operator and, for $v(x) = (v_1(x), \ldots, v_N(x)$, $\operatorname{div} v(x) = \sum_{i=1}^N \frac{\partial v}{\partial x_i}(x)$ is the usual divergence operator. If u is a radially symmetric function, i.e., $u(x) = y(t)$, $t = \|x\|$, $\|\cdot\|$ being the Euclidean norm in $\mathbb{R}^N$, the (partial) differential operator Δ_α can be reduced to the ordinary differential operator

$$\Delta_\alpha u(x) = t^{1-N}\left(t^{N-1}\Phi(y'(t))\right)', \quad ' = \frac{\mathrm{d}}{\mathrm{d}t}.$$

There are at least another two ways how half-linear equations can be understood. First, as a natural generalization of second order linear differential equations. Indeed, if $\alpha = 2$, then (3.1) reduces to (2.9). Second, as a special case of certain Emden-Fowler type equations, which are studied in the next chapter.

Some of the results will be formulated for the half-linear equation in a more special form

$$(\Phi(y'))' + p(t)\Phi(y) = 0. \tag{3.2}$$

An important role in some of the proofs will be played (not surprisingly) by the generalized Riccati differential equation

$$w' + p(t) + (\alpha - 1)|w|^\beta = 0, \tag{3.3}$$

which is associated to (3.2) by the substitution $w = \Phi(y'/y)$. Note that the Riccati technique is extremely useful also in many other parts of qualitative theory of half-linear equations, see Došlý, Řehák [28].

Basic classification and asymptotics of nonoscillatory solutions to half-linear equations are discussed e.g. in Cecchi, Došlá, Marini [19] and Chanturiya [20], see also [28, Chapter 4].

The results in this chapter can be understood as a half-linear extension of some of the results from the previous chapter, but there are also some original formulas. However we should emphasize that due to the lack of the additivity of the solutions space of equation (3.1), many steps in the proofs (if not the proof entire) require a quite new approach or at least a highly nontrivial modification comparing with the linear case. As an example of the problematic point we can mention that there is no reduction of order formula for (3.1), and so we cannot so simply construct a linearly independent solution provided one solution (with some known properties) is at disposal.

3.2 $\mathcal{RV}$ and $\mathcal{RB}$ solutions of half-linear equations

The results of this section are based on the paper [59] by Jaroš, Kusano, and Tanigawa.

3.2.1 Auxiliary statement

Proposition 3.1. *Put $P(t) = \int_t^\infty p(s)\,\mathrm{d}s$ and suppose that there exists a continuous function $h : [t_0,\infty) \to (0,\infty)$, $t_0 \ge 0$, such that $\lim_{t\to\infty} h(t) = 0$, $|P(t)| \le h(t)$, $t \ge t_0$, and*

$$\int_t^\infty h^\beta(s)\,\mathrm{d}s \le \frac{1}{\alpha-1}a^{\beta-1}h(t), \quad t \ge t_0, \tag{3.4}$$

for some positive constant

$$a < \frac{1}{\alpha}\left(\frac{\alpha-1}{\alpha}\right)^{\alpha-1}. \tag{3.5}$$

Then equation (3.2) *is nonoscillatory and has a solution of the form*

$$y(t) = \exp\left\{\int_{t_0}^{t} \Phi^{-1}[v(s) + P(s)]\,\mathrm{d}s\right\}, \quad t \geq t_0, \tag{3.6}$$

where v is a solution of the integral equation

$$v(t) = (\alpha - 1)\int_t^\infty |v(s) + P(s)|^\beta \mathrm{d}s, \quad t \geq t_0, \tag{3.7}$$

satisfying

$$v(t) = O(P(t)) \ \text{ as } \ t \to \infty. \tag{3.8}$$

Proof. Consider the function y defined by (3.6). Recall that if w is a solution of generalized Riccati equation (3.3), then $\exp\left\{\int_{t_0}^{t}\Phi^{-1}(w(s))\,ds\right\}$ is a (nonoscillatory) solution of (3.2). Hence, y is a solution of (3.2) if v is chosen in such a way that $w = v + P$ satisfies (3.3) on $[t_0, \infty)$. The differential equation for v then reads $v' + (\alpha-1)|v + P(t)|^\beta = 0$, which upon integration under the additional requirement that $\lim_{t\to\infty} v(t) = 0$, yields (3.7). We denote by $C_h[t_0, \infty)$ the set of all continuous functions v on $[t_0, \infty)$ such that

$$\|v\|_h = \sup_{t \geq t_0} \frac{|v(t)|}{h(t)} < \infty.$$

Clearly, $C_h[t_0, \infty)$ is a Banach space equipped with the norm $\|v\|_h$. Let Ω be a subset of $C_h[t_0, \infty)$ defined by $\Omega = \{v \in C_h[t_0, \infty) : |v(t)| \leq (\alpha - 1)h(t), t \geq t_0\}$ and define the mapping $\mathcal{T} : \Omega \to C_h[t_0, \infty)$ by

$$\mathcal{T}v(t) = (\alpha - 1)\int_t^\infty |v(s) + P(s)|^\beta \mathrm{d}s, \quad t \geq t_0. \tag{3.9}$$

If $v \in \Omega$, then

$$|\mathcal{T}v(t)| \leq (\alpha - 1)\alpha^\beta \int_t^\infty h^\beta(s)\,\mathrm{d}s \leq \alpha^\beta a^{\beta-1} h(t),$$

$t \geq t_0$, which implies that

$$\|\mathcal{T}v\|_h \leq \alpha^\beta a^{\beta-1} < \alpha^\beta \left(\frac{(\alpha-1)^{\alpha-1}}{\alpha^\alpha}\right)^{\beta-1} = \alpha - 1. \tag{3.10}$$

Thus $\mathcal{T}$ maps Ω into itself. If $v_1, v_2 \in \Omega$, then, using the Mean Value Theorem, we see that

$$\begin{aligned}
|\mathcal{T}v_1(t) - \mathcal{T}v_2(t)| &\leq (\alpha-1)\int_t^\infty \left||v_1(s)+P(s)|^\beta - |v_2(s)+P(s)|^\beta\right| \mathrm{d}s \\
&\leq (\alpha-1)\beta\int_t^\infty (\alpha h(s))^{\beta-1}|v_1(s)-v_2(s)|\,\mathrm{d}s \\
&= \alpha^\beta \int_t^\infty h^\beta(s)\frac{|v_1(s)-v_2(s)|}{h(s)}\,\mathrm{d}s \\
&\leq \alpha^\beta(\beta-1)a^{\beta-1}h(t)\|v_1-v_2\|_h,
\end{aligned}$$

$t \geq t_0$, from which it follows that

$$\|\mathcal{T}v_1 - \mathcal{T}v_2\|_h \leq (\beta-1)\alpha^\beta a^{\beta-1}\|v_1-v_2\|_h.$$

In view of (3.5) (or (3.10)), this implies that $\mathcal{T}$ is a contraction mapping on Ω. Therefore, by the contraction mapping principle, there exists an element $v \in \Omega$ such that $v = \mathcal{T}v$, that is, a solution of integral equation (3.7). Thus the function $y(t)$ defined by (3.6) with this $v(t)$ gives a solution of (3.2) on $[t_0, \infty)$. The fact that v satisfies (3.8) is a consequence of $v \in \Omega$. This completes the proof. □

3.2.2 $\mathcal{RV}$ solutions with different indices

The following theorem is a generalization of the main part of Theorem 2.2. Proposition 3.1 is applied in the proof.

Theorem 3.1. *Equation* (3.2) *is nonoscillatory and and has two solutions y_1 and y_2 such that $y_1 \in \mathcal{NSV}$, $y_2 \in \mathcal{NRV}(1)$ if and only if*

$$\lim_{t\to\infty} t^{\alpha-1}\int_t^\infty p(s)\,\mathrm{d}s = 0 \tag{3.11}$$

Proof. If. Suppose that the condition from the theorem holds. Put

$$\varphi(t) = \sup_{s\geq t}\left|s^{\alpha-1}\int_s^\infty p(\tau)\,\mathrm{d}\tau\right|. \tag{3.12}$$

Then φ is nonincreasing and tends to zero as $t \to \infty$. Let $t_0 > 0$ be such that

$$\varphi(t) < \frac{1}{\alpha}\left(\frac{\alpha-1}{\alpha}\right)^{\alpha-1} \quad\text{and}\quad |P(t)| \leq \frac{\varphi(t)}{t^{\alpha-1}}$$

for $t \geq t_0$. Put $h(t) = \varphi(t)t^{1-\alpha}$. Then $|P(t)| \leq h(t)$ holds and

$$\int_t^\infty h^\beta(s)\,ds = \int_t^\infty \left(\frac{\varphi(s)}{s^{\alpha-1}}\right)^\beta \mathrm{d}s \leq \frac{\varphi^\beta(t)}{(\alpha-1)t^{\alpha-1}} = \frac{1}{\alpha-1}\varphi^{\beta-1}(t)h(t)$$

$t \geq t_0$. Consequently, by Proposition 3.1, (3.2) has a nonoscillatory solution of the form (3.6) on $[t_0, \infty)$ with v satisfying (3.8). Since

$$t^{\alpha-1}v(t) = O(t^{\alpha-1}h(t)) = o(1) \text{ and } t^{\alpha-1}P(t) = O(t^{\alpha-1}h(t)) = o(1)$$

as $t \to \infty$, we conclude that y is a normalized slowly varying function. The existence of a solution $y_2 \in \mathcal{NRV}(1)$ follows from the proof of Theorem 3.2.

Only if. It follows from the only if part of the proof of Theorem 3.2 □

The following theorem is a generalization of the main part of Theorem 2.3. As it can be easily seen, the previous theorem can be included in its statement, but we prefer to distinguish these two results because the part dealing with the existence of a $\mathcal{SV}$ solution uses different ideas. In fact, in the proof of Theorem 3.2 we consider only the nonzero roots of the associated algebraic equation, since the case with the zero root is treated in the proof of the previous theorem.

Let λ_1 and λ_2, $\lambda_1 < \lambda_2$, denote the two real roots of the equation

$$|\lambda|^\beta - \lambda + A = 0. \tag{3.13}$$

It is easy to see that (3.13) has two distinct real roots if and only if $A < \frac{1}{\alpha}\left(\frac{\alpha-1}{\alpha}\right)^{\alpha-1}$. Clearly, $\lambda_1 < 0 < \lambda_2$ if $A < 0$, and $0 < \lambda_1 < \lambda_2$ if $0 < A < \frac{1}{\alpha}\left(\frac{\alpha-1}{\alpha}\right)^{\alpha-1}$. It should be noticed that $\alpha\Phi^{-1}(\lambda_1) < \alpha - 1 < \alpha\Phi^{-1}(\lambda_2)$.

Theorem 3.2. *Equation* (3.2) *is nonoscillatory and has two solutions y_1 and y_2 such that $y_1 \in \mathcal{NRV}(\Phi^{-1}(\lambda_1))$ and $y_2 \in \mathcal{NRV}(\Phi^{-1}(\lambda_2))$ if and only if*

$$\lim_{t\to\infty} t^{\alpha-1}\int_t^\infty p(s)\,\mathrm{d}s = A \in \left(-\infty, \frac{1}{\alpha}\left(\frac{\alpha-1}{\alpha}\right)^{\alpha-1}\right). \tag{3.14}$$

Proof. Only if. Let y_i be solutions belonging to $\mathcal{NRV}(\Phi^{-1}(\lambda_i))$, $i = 1, 2$. From the representation theorem it follows that

$$\lim_{t\to\infty} t\frac{y_i'(t)}{y_i(t)} = \Phi^{-1}(\lambda_i), \text{ so that } \lim_{t\to\infty}\frac{y_i'(t)}{y_i(t)} = 0, \quad i = 1, 2. \tag{3.15}$$

Put $w_i = \Phi(y'/y)$, $i = 1, 2$. Then w_i satisfies generalized Riccati equation (3.3), from which, integrating on $[t, \infty)$ and noting that $\lim_{t\to\infty} w_i(t) = 0$, we have

$$t^{\alpha-1}w_i(t) = (\alpha-1)t^{\alpha-1}\int_t^\infty \frac{|s^{\alpha-1}w_i(s)|^\beta}{s^\alpha}\,\mathrm{d}s + t^{\alpha-1}\int_t^\infty p(s)\,\mathrm{d}s, \quad i = 1, 2, \tag{3.16}$$

for all sufficiently large t. Let $t \to \infty$ in (3.16). Using (3.15), we conclude that

$$\lim_{t\to\infty} t^{\alpha-1}\int_t^\infty p(s)\,\mathrm{d}s = \lambda_i - |\lambda_i|^\beta = A, \quad i = 1, 2.$$

If. Assume that (3.14) holds. Put $\omega(t) = t^{\alpha-1}\int_t^\infty p(s)\,\mathrm{d}s - A$ and consider the functions

$$y_i(t) = \exp\left\{\int_{t_i}^t \Phi^{-1}\left(\frac{\lambda_i + \omega(s) + v_i(s)}{s^{\alpha-1}}\right)\mathrm{d}s\right\}, \quad i = 1, 2. \tag{3.17}$$

Then the function y_i is a solution of (3.2) on $[t_i, \infty)$ if v_i is chosen in such a way that $w_i = (\lambda_i + \omega + v_i)/t^{\alpha-1}$ satisfies (3.3) on $[t_i, \infty)$, $i = 1, 2$. The differential equation for v_i then reads

$$v_i' - \frac{\alpha-1}{t}v_i + \frac{\alpha-1}{t}\left(|\lambda_i + \omega(t) + v_i|^\beta - |\lambda_i|^\beta\right) = 0, \quad i = 1, 2. \tag{3.18}$$

We rewrite (3.18) as

$$v_i' + \frac{\alpha\Phi^{-1}(\lambda_i + \omega(t)) - (\alpha-1)}{t}v_i + \frac{\alpha-1}{t}\left[|\lambda_i + \omega(t) + v_i|^\beta - \beta\Phi^{-1}(\lambda_i + \omega(t))v_i - |\lambda_i|^\beta\right] = 0$$

and transform it into

$$(r_i(t)v_i)' + \frac{\alpha-1}{t}r_i(t)F_i(t, v_i) = 0, \tag{3.19}$$

where

$$r_i(t) = \exp\left\{\int_1^t \frac{\alpha\Phi^{-1}(\lambda_i + \omega(s)) - (\alpha-1)}{s}\,\mathrm{d}s\right\}$$

and

$$F_i(t, v) = |\lambda_i + \omega(t) + v|^\beta - \beta\Phi^{-1}(\lambda_i + \omega(t))v - |\lambda_i|^\beta, \quad i = 1, 2.$$

It is convenient to express $F_i(t, v)$ as $F_i(t, v) = G_i(t, v) + z_i(t)$, with $G_i(t, v)$ and $z_i(t)$ defined by

$$G_i(t, v) = |\lambda_i + \omega(t) + v|^\beta - \beta\Phi^{-1}(\lambda_i + \omega(t))v - |\lambda_i + \omega(t)|^\beta$$

and $z_i(t) = |\lambda_i + \omega(t)|^\beta - |\lambda_i|^\beta$, $i = 1, 2$. Now we suppose that $A \neq 0$ in (3.14), which implies $\lambda_i \neq 0$ for $i = 1, 2$. Let $t_0 > 0$ be such that $|\omega(t)| \le |\lambda_i|/4$ for $t \ge t_0$, $i = 1, 2$. This is possible because $\omega(t) \to 0$ as $t \to \infty$ by hypothesis. It follows that $\frac{3}{4}|\lambda_i| \le |\lambda_i + \omega(t)| \le \frac{5}{4}|\lambda_i|$ for $t \ge t_0$, $i = 1, 2$. We observe that there exist positive constants $K_i(\alpha), L_i(\alpha)$ and $M_i(\alpha)$ such that $|G_i(t, v)| \le K_i(\alpha)v^2$,

$$\left|\frac{\partial G_i}{\partial v}(t, v)\right| \le L_i(\alpha)|v| \tag{3.20}$$

and $|z_i(t)| \le M_i(\alpha)|\omega(t)|$ for $t \ge t_0$ and $|v| \le |\lambda_i|/4$, $i = 1, 2$. In fact, the last two estimations follow from the Mean Value Theorem, while the estimation for G_i is a consequence of the L'Hospital rule applied to G_i:

$$\lim_{v\to 0}\frac{G_i(t, v)}{v^2} = \frac{1}{2}\lim_{v\to 0}\frac{\partial^2 G_i(t, v)}{\partial v^2} = \frac{\beta}{2(\alpha-1)}|\lambda_i + \omega(t)|^{\beta-2}.$$

Let us examine equation (3.19) with $i = 1$. The following properties of r_1 are needed: $r_1 \in \mathcal{NRV}(\Phi^{-1}(\alpha) - (\alpha - 1))$, $\lim_{t\to\infty} r_1(t) = 0$,

$$\lim_{t\to\infty} \frac{\alpha - 1}{r_1(t)} \int_t^\infty \frac{r_1(s)}{s}\,\mathrm{d}s = (\alpha - 1)\lim_{t\to\infty} \frac{-r_1(t)}{tr_1'(t)} = \frac{\alpha - 1}{\alpha - 1 - \alpha\Phi^{-1}(\lambda_1)}, \tag{3.21}$$

$$\lim_{t\to\infty} \frac{\alpha - 1}{r_1(t)} \int_t^\infty \frac{r_1(s)}{s} z(s)\,\mathrm{d}s = 0 \ \text{ if } \ z \in C[0,\infty), \ \text{ and } \ \lim_{t\to\infty} z(t) = 0. \tag{3.22}$$

Let ε_1 be a positive constant such that $\varepsilon_1 < \min\{1, |\lambda_1|/4\}$ and

$$\frac{2(\alpha - 1)}{\alpha - 1 - \alpha\Phi^{-1}(\lambda_1)}[K_1(\alpha) + L_1(\alpha) + M_1(\alpha)]\varepsilon_1 \le 1, \tag{3.23}$$

and choose $t_1 \ge t_0$ so that

$$|\omega(t)| \le \varepsilon_1^2, \ \ t \ge t_1, \tag{3.24}$$

and

$$\frac{\alpha - 1}{r_1(t)} \int_t^\infty \frac{r_1(s)}{s}\,\mathrm{d}s \le \frac{2(\alpha - 1)}{\alpha - 1 - \alpha\Phi^{-1}(\lambda_1)}, \ \ t \ge t_1. \tag{3.25}$$

Note that (3.25) is an immediate consequence of (3.21).

Let $C_0[t_1,\infty)$ denote the set of all continuous functions on $[t_1,\infty)$ which tend to zero as $t \to \infty$. Then $C_0[t_1,\infty)$ is a Banach space with the sup-norm $\|v\| = \sup\{|v(t)| : t \ge t_1\}$. Consider the set $\Omega_1 \subset C_0[t_1,\infty)$ defined by $\Omega_1 = \{v \in C_0[t_1,\infty) : |v(t)| \le \varepsilon_1, t \ge t_1\}$ and define the integral operator $\mathcal{T}_1$ by

$$(\mathcal{T}_1 v)(t) = \frac{\alpha - 1}{r_1(t)} \int_t^\infty \frac{r_1(s)}{s} F_1(s, v(s))\,\mathrm{d}s,$$

$t \ge t_1$. It can be shown that $\mathcal{T}_1$ is a contraction mapping on Ω_1. In fact, if $v \in \Omega_1$, then using the above inequalities we see that

$$\begin{aligned}
|(\mathcal{T}_1 v)(t)| &\le \frac{\alpha - 1}{r_1(t)} \int_t^\infty \frac{r_1(s)}{s}(|G_1(s, v(s))| + |z_1(s)|)\,\mathrm{d}s \\
&\le \frac{\alpha - 1}{r_1(t)} \int_t^\infty \frac{r_1(s)}{s}(K_1(\alpha)v^2(s) + M_1(\alpha)|\omega(s)|)\,\mathrm{d}s \\
&\le \frac{\alpha - 1}{r_1(t)} \int_t^\infty \frac{r_1(s)}{s}(K_1(\alpha) + M_1(\alpha))\varepsilon_1^2\,\mathrm{d}s \\
&\le \frac{2(\alpha - 1)}{\alpha - 1 - \alpha\Phi^{-1}(\lambda_1)}[K_1(\alpha) + M_1(\alpha)]\varepsilon_1^2,
\end{aligned}$$

$t \ge t_1$. Since $F_1(t, v(t)) \to 0$ as $t \to \infty$, we have $\lim_{t\to\infty}(\mathcal{T}_1 v)(t) = 0$ by (3.22). It follows that $\mathcal{T}_1 v \in \Omega_1$, and so $\mathcal{T}_1$ maps Ω_1 into itself. Furthermore, if $u, v \in \Omega_1$, then using (3.20) and (3.23), we obtain

$$\begin{aligned}|(\mathcal{T}_1 v)(t) - (\mathcal{T}_1 u)(t)| &\leq \frac{\alpha-1}{r_1(t)}\int_t^\infty \frac{r_1(s)}{s}|F_1(s,v(s)) - F_1(s,u(s))|\,\mathrm{d}s \\ &= \frac{\alpha-1}{r_1(t)}\int_t^\infty \frac{r_1(s)}{s}|G_1(s,v(s)) - G_1(s,u(s))|\,\mathrm{d}s \\ &\leq \frac{2(\alpha-1)}{\alpha-1-\alpha\Phi^{-1}(\lambda_1)}L_1(\alpha)\varepsilon_1\|v-u\|,\end{aligned}$$

$t \geq t_1$, which implies that

$$\|\mathcal{T}_1 v - \mathcal{T}_1 u\| \leq \frac{2(\alpha-1)L_1(\alpha)}{\alpha-1-\alpha\Phi^{-1}(\lambda_1)}\varepsilon_1\|v-u\|.$$

In view of (3.23), this shows that $\mathcal{T}_1$ is a contraction mapping on Ω_1. The contraction mapping principle then ensures the existence of a unique fixed element $v_1 \in \Omega_1$ such that $v_1 = \mathcal{T}_1 v_1$, which is equivalent to the integral equation

$$v_1(t) = \frac{\alpha-1}{r_1(t)}\int_t^\infty \frac{r_1(s)}{s}F_1(s,v_1(s))\,\mathrm{d}s, \tag{3.26}$$

$t \geq t_1$. Differentiation of (3.26) shows that v_1 satisfies differential equation (3.19) with $i = 1$ on $[t_1,\infty)$ and so substitution of this v_1 into (3.17) gives rise to a solution y_1 of half-linear equation (3.2) defined on $[t_1,\infty)$. Since $\lim_{t\to\infty} v_1(t) = 0$, from the representation theorem we have $y_1 \in \mathcal{NRV}(\Phi^{-1}(\lambda_1))$.

Our next task is to solve equation (3.19) for $i = 2$ in order to construct a larger solution y_2 of (3.2) via formula (3.17). It is easy to see that $r_2 \in \mathcal{NRV}(\alpha\Phi^{-1}(\lambda_2) - \alpha + 1)$, $\lim_{t\to\infty} r_2(t) = \infty$ and that for any fixed $t_2 > 0$,

$$\lim_{t\to\infty}\frac{\alpha-1}{r_2(t)}\int_{t_2}^t \frac{r_2(s)}{s}\,\mathrm{d}s = (\alpha-1)\lim_{t\to\infty}\frac{r_2(t)}{tr_2'(t)} = \frac{\alpha-1}{\alpha\Phi^{-1}(\lambda_2)-\alpha+1},$$

$$\lim_{t\to\infty}\frac{\alpha-1}{r_2(t)}\int_{t_2}^t \frac{r_2(s)z(s)}{s}\,\mathrm{d}s = 0 \ \text{ if } \ z \in C[t_2,\infty) \ \text{ and } \ \lim_{t\to\infty} z(t) = 0.$$

Let $\varepsilon_2 > 0$ be small enough so that

$$\frac{2(\alpha-1)}{\alpha\Phi^{-1}(\lambda_2)-\alpha+1}[K_2(\alpha) + L_2(\alpha) + M_2(\alpha)]\varepsilon_2 \leq 1,$$

and choose $t_2 > 0$ so large that $\omega(t) \leq \varepsilon_2^2$, $t \geq t_2$, and

$$\frac{\alpha-1}{r_2(t)}\int_{t_2}^t \frac{r_2(s)}{2}\,\mathrm{d}s \leq \frac{2(\alpha-1)}{\alpha\Phi^{-1}(\lambda_2)-\alpha+1},$$

$t \geq t_2$. Define the set $\Omega_2 \subset C_0[t_2,\infty)$ and the integral operator $\mathcal{T}_2$ by $\Omega_2 = \{v \in C_0[t_2,\infty) : |v(t)| \leq \varepsilon_2, t \geq t_2\}$, and

$$(\mathcal{T}_2 v)(t) = -\frac{\alpha-1}{r_2(t)}\int_{t_2}^t \frac{r_2(s)}{s}F_2(s,v(s))\,\mathrm{d}s,$$

$t \geq t_2$. It is a matter of easy calculation to verify that $\mathcal{T}_2$ is a contraction mapping on Ω_2. Therefore there exists a unique fixed element $v_2 \in \Omega_2$ of $\mathcal{T}_2$, which satisfies the integral equation

$$v_2(t) = -\frac{\alpha - 1}{r_2(t)} \int_{t_2}^{t} \frac{r_2(s)}{s} F(s, v_2(s))\, \mathrm{d}s,$$

$t \geq t_2$, and hence differential equation (3.19) with $i = 2$. Then the function y_2 defined by (3.17) with this v_2 is a nonoscillatory solution of (3.2) on $[t_2, \infty)$. The fact that $y_2 \in \mathcal{NRV}(\Phi^{-1}(\lambda_2))$ follows from the representation theorem. This finishes the proof of the "if" part of the theorem for the case $A \neq 0$. If $A = 0$, then equation (3.13) has the two real roots $\lambda_1 = 0$, $\lambda_2 = 1$. The solution $y_1 \in \mathcal{NRV}(0) = \mathcal{NSV}$ of (3.2) corresponding to λ_1 has already been constructed in Theorem 3.1. The existence of the solution $y_2 \in \mathcal{NRV}(1)$ corresponding to λ_2 can be proved in exactly the same manner as developed for the case $A \neq 0$. □

3.2.3 $\mathcal{RV}$ solutions in the border case

Let us consider equation (3.2) for which the condition

$$\lim_{t\to\infty} t^{\alpha-1} \int_t^{\infty} p(s)\, \mathrm{d}s = \frac{1}{\alpha}\left(\frac{\alpha-1}{\alpha}\right)^{\alpha-1} \tag{3.27}$$

is satisfied. Such an equation can be regarded as a perturbation of the generalized Euler equation

$$(\Phi(y'))' + \frac{\gamma}{t^{\alpha}}\Phi(y) = 0 \tag{3.28}$$

with $\gamma = \bar{\gamma} = \left(\frac{\alpha-1}{\alpha}\right)^{\alpha}$. Although (3.28) is nonoscillatory, because it has a solution $y(t) = t^{(\alpha-1)/\alpha}$, its perturbation may be oscillatory or nonoscillatory depending on the asymptotic behavior of the perturbed term as $t \to \infty$, see Došlý, Řehák [28]. Our purpose here is to show the existence of a class of perturbations which preserve the nonoscillation character of (3.28). The result can be seen as a generalization of Theorem 2.4.

Theorem 3.3. *Suppose that* (3.27) *holds. Put*

$$\Upsilon(t) = t^{\alpha-1} \int_t^{\infty} p(s)\, \mathrm{d}s - \frac{1}{\alpha}\left(\frac{\alpha-1}{\alpha}\right)^{\alpha-1}$$

and suppose that

$$\int^{\infty} \frac{|\Upsilon(t)|}{t}\, \mathrm{d}t < \infty \tag{3.29}$$

and

$$\int^{\infty} \frac{\Psi(t)}{t}\, \mathrm{d}t < \infty, \quad \textit{where} \quad \Psi(t) = \int_t^{\infty} \frac{|\Upsilon(s)|}{s}\, \mathrm{d}s. \tag{3.30}$$

Then equation (3.2) *is nonoscillatory and has a normalized regularly varying solution with index* $(\alpha-1)/\alpha$ *of the form* $y(t)=t^{(\alpha-1)/\alpha}L(t)$ *with* $L\in\mathcal{NSV}$ *and* $\lim_{t\to\infty}L(t)=\ell\in(0,\infty)$.

Proof. The solution is sought in the form

$$y(t)=\exp\int_T^t\Phi^{-1}\left(\frac{\bar\gamma+\Upsilon(s)+v(s)}{s^{\alpha-1}}\right)\mathrm{d}s,\quad \bar\gamma=\left(\frac{\alpha-1}{\alpha}\right)^{\alpha},\tag{3.31}$$

for some $T>0$ and $v:[T,\infty)\to\mathbb{R}$. The same argument as in the proof of the "if" part of the previous theorem leads to the differential equation for v

$$(r(t)v)'+\frac{\alpha-1}{t}r(t)F(t,v)=0,\tag{3.32}$$

where

$$r(t)=\exp\left(\int_1^t\frac{\alpha\Phi^{-1}(\bar\gamma+\Upsilon(s))-\alpha+1}{s}\,\mathrm{d}s\right)$$

and

$$F(t,v)=|\bar\gamma+\Upsilon(t)+v|^{\beta}-\beta\Phi^{-1}(\bar\gamma+\Upsilon(t))v-\bar\gamma^{\beta}.\tag{3.33}$$

Choose $t_0>0$ so that

$$|\Upsilon(t)|\le\frac{\bar\gamma}{4},\tag{3.34}$$

$t\ge t_0$. Since

$$|\Phi^{-1}(\bar\gamma+\Upsilon(t))-\alpha+1|=\alpha|(\bar\gamma+\Upsilon(t))^{\beta-1}-\bar\gamma^{\beta-1}|\le\alpha m(\alpha)|\Upsilon(t)|,$$

$t\ge t_0$, for some constant $m(\alpha)>0$, we see in view of (3.29) that r is a slowly varying function and tends to a finite positive limit as $t\to\infty$. It follows that there exists $t_1\ge t_0$ such that

$$r(s)/r(t)\le 2\tag{3.35}$$

for $s\ge t\ge t_1$. We rewrite the function $F(t,v)$ defined by (3.33) as $F(t,v)=G(t,v)+z(t)$, where

$$G(t,v)=|\bar\gamma+\Upsilon(t)+v|^{\beta}-\beta\Phi^{-1}(\bar\gamma+\Upsilon(t))v-|\bar\gamma+\Upsilon(t)|^{\beta}$$

and $z(t)=|\bar\gamma+\Upsilon(t)|^{\beta}-\bar\gamma^{\beta}$. As it is easily seen, there exist positive constants $K(\alpha)$, $L(\alpha)$ and $M(\alpha)$ such that

$$|G(t,v)|\le K(\alpha)v^2,\tag{3.36}$$

$$\left|\frac{\partial G(t,v)}{\partial v}\right|\le L(\alpha)|v|\tag{3.37}$$

and $|h(t)|\le M(\alpha)|\Upsilon(t)|$ for $t\ge t_1$ and $|v|\le\bar\gamma/4$. Let $T>t_1$ be large enough so that

$$4(p-1)M(\alpha)\Psi(t)\le\frac{\bar\gamma}{4},\tag{3.38}$$

$t \geq T$,

$$16(\alpha - 1)^2 K(\alpha) M(\alpha) \int_T^\infty \frac{\Psi(s)}{s} \, \mathrm{d}s \leq 1,$$

and

$$16(\alpha - 1)^2 L(\alpha) M(\alpha) \int_T^\infty \frac{\Psi(s)}{s} \, ds \leq 1. \tag{3.39}$$

We want to solve the integral equation

$$v(t) = \frac{\alpha - 1}{r(t)} \int_t^\infty \frac{r(s)}{s} F(s, v(s)) \, \mathrm{d}s, \tag{3.40}$$

$t \geq T$, which follows from (3.32), subject to the condition $\lim_{t \to \infty} v(t) = 0$. Let $C_\Psi[T, \infty)$ denote the set of all continuous functions v on $[T, \infty)$ such that

$$\|v\|_\Psi = \sup_{t \geq T} \frac{|v(t)|}{\Psi(t)} < \infty.$$

Clearly, $C_\Psi[T, \infty)$ is a Banach space equipped with the norm $\|v\|_\Psi$. Consider the set $\Omega \subset C_\Psi[T, \infty)$ and the mapping $\mathcal{T}\Omega \to C_\Psi[T, \infty)$ defined by

$$\Omega = \{v \in C_\Psi[T, \infty) \, : \, |v(t)| \leq 4(\alpha - 1) M(\alpha) \Psi(t), \, t \geq T\} \tag{3.41}$$

and

$$\mathcal{T} v(t) = \frac{\alpha - 1}{r(t)} \int_t^\infty \frac{r(s)}{s} F(s, v(s)) \, \mathrm{d}s = \frac{\alpha - 1}{r(t)} \int_t^\infty \frac{r(s)}{s} [G(s, v(s)) + z(s)] \, \mathrm{d}s,$$

$t \geq T$. Using (3.35), (3.36) and (3.37), we see that

$$\frac{\alpha - 1}{r(t)} \int_t^\infty \frac{r(s)}{s} |z(s)| \, \mathrm{d}s \leq 2(\alpha - 1) \int_t^\infty \frac{M(\alpha) |\Upsilon(s)|}{s} \, \mathrm{d}s = 2(\alpha - 1) M(\alpha) \Psi(t),$$

$t \geq T$, and that

$$\begin{aligned}
\frac{\alpha - 1}{r(t)} \int_t^\infty \frac{r(s)}{s} |G(s, v(s))| \, \mathrm{d}s &\leq 2(\alpha - 1) \int_t^\infty \frac{K(\alpha)[4(\alpha - 1) M(\alpha) \Psi(s)]^2}{s} \, \mathrm{d}s \\
&= 32(\alpha - 1)^3 K(\alpha) M^2(\alpha) \int_t^\infty \frac{\Psi^2(s)}{s} \, \mathrm{d}s \\
&\leq 32(\alpha - 1)^3 K(\alpha) M^2(\alpha) \Psi(t) \int_t^\infty \frac{\Psi(s)}{s} \, \mathrm{d}s \\
&\leq 32(\alpha - 1)^3 K(\alpha) M^2(\alpha) \Psi(t) \int_T^\infty \frac{\Psi(s)}{s} \, \mathrm{d}s \leq 2(\alpha - 1) M(\alpha) \Psi(t),
\end{aligned}$$

$t \geq T$. This shows that $v \in \Omega$ implies $\mathcal{T}v \in \Omega$, and hence $\mathcal{T}$ maps Ω into itself. If $u, v \in \Omega$, then using (3.37) we have

$$\begin{aligned}
|\mathcal{T}v(t) - \mathcal{T}u(t)| &\leq \frac{\alpha-1}{r(t)} \int_t^\infty \frac{r(s)}{s} |G(s, v(s)) - G(s, u(s))| \, \mathrm{d}s \\
&\leq 2(\alpha-1) \int_t^\infty \frac{4(\alpha-1)L(\alpha)M(\alpha)\Psi(s)|v(s)-u(s)|}{s} \, \mathrm{d}s \\
&= 8(\alpha-1)^2 L(\alpha)M(\alpha) \int_t^\infty \frac{\Psi^2(s)|v(s)-u(s)|}{s\Psi(s)} \, \mathrm{d}s \\
&\leq 8(\alpha-1)^2 L(\alpha)M(\alpha)\Psi(t)\|v-u\|_\Psi \int_t^\infty \frac{\Psi(s)}{s} \, \mathrm{d}s,
\end{aligned}$$

$t \geq T$, from which, in view of (3.39), we conclude that $\mathcal{T}$ is a contraction mapping: $\|\mathcal{T}v - \mathcal{T}u\|_\Psi \leq \|v-u\|_\Psi/2$. Let $v \in \Omega$ be a unique fixed element of $\mathcal{T}$. Then v satisfies (3.40), and hence (3.32), on $[T, \infty)$, and the function y defined by (3.31) provides a nonoscillatory solution of (3.2) on $[T, \infty)$. Since $\Phi^{-1}(\bar{\gamma} + \Upsilon(t) + v(t)) \to (\alpha-1)/p$ as $t \to \infty$, $y \in \mathcal{NRV}((\alpha-1)/\alpha)$, and y is expressed as $y(t) = t^{(\alpha-1)/\alpha} L(t)$, where

$$L(t) = \exp\left\{ \int_1^t \frac{\Phi^{-1}(\bar{\gamma} + \Upsilon(s) + v(s)) - \bar{\gamma}^{\beta-1}}{s} \, \mathrm{d}s \right\},$$

$t \geq T$. Noting that $|\Upsilon(t) + v(t)| \leq \bar{\gamma}/2$, $t \geq T$, by (3.34) and (3.38), and applying the Mean Value Theorem, we see with the use of (3.41) that $|\Phi^{-1}(\bar{\gamma} + \Upsilon(t) + v(t)) - \bar{\gamma}^{\beta-1}| \leq N(\alpha)(|\Upsilon(t)| + |\Psi(t)|)$, $t \geq T$, for some constant $N(\alpha) > 0$. This, combined with the hypotheses (3.29) and (3.30), guarantees that $L(t)$ tends to a finite positive limit as $t \to \infty$. □

3.2.4 $\mathcal{RB}$ solutions

In the next theorem we somehow relax the condition on the existence of the limit in (3.11) and (3.14), and the existence of a $\mathcal{RB}$ solution follows; an important role in the proof is played by Proposition 3.1. The result generalizes Theorem 2.5. But observe that here we have guaranteed the existence of (at least) one $\mathcal{RB}$ solution, while in the linear case all eventually positive solutions are $\mathcal{RB}$. As already indicated above, the reason is that here we cannot use a reduction of order formula and the fact that any solution is a linear combination of elements of the fundamental set.

Theorem 3.4. *If*

$$-\frac{1}{\alpha}\left(\frac{\alpha-1}{\alpha}\right)^{\alpha-1} < \liminf_{t\to\infty} t^{\alpha-1} \int_t^\infty p(s) \, \mathrm{d}s \leq \limsup_{t\to\infty} t^{\alpha-1} \int_t^\infty p(s) \, \mathrm{d}s < \frac{1}{\alpha}\left(\frac{\alpha-1}{\alpha}\right)^{\alpha-1}$$

holds, then equation (3.2) *is nonoscillatory and has a normalized regularly bounded solution.*

Proof. If the assumption holds, then there exist positive constants t_0 and a satisfying (3.5) such that $|t^{\alpha-1}P(t)| \le a$, $t \ge t_0$. Put $h(t) = at^{1-\alpha}$. Then it can be easily verified that h satisfies $|P(t)| \le h(t)$ and (3.4). So, by Proposition 3.1, (3.2) has a nonoscillatory solution of the form

$$\begin{aligned} y(t) &= \exp\left\{\int_{t_0}^{t} \Phi^{-1}(v(s) + P(s))\,\mathrm{d}s\right\} \\ &= \exp\left\{\int_{t_0}^{t} \frac{\Phi^{-1}[s^{\alpha-1}(v(s) + P(s))]}{s}\,\mathrm{d}s\right\}, \end{aligned} \tag{3.42}$$

$t \ge t_0$, with v satisfying (3.8). Since

$$\left|\Phi^{-1}[t^{\alpha-1}(v(t) + \sigma(t))]\right| \le \alpha^{\beta-1}(t^{\alpha-1}h(t))^{\beta-1} = (a\alpha)^{\beta-1},$$

$t \ge t_0$, the solution y is a normalized $\mathcal{RB}$ function by the representation theorem. □

3.3 More precise information about $\mathcal{SV}$ solutions

The observations in this section are based on the paper [99] by Kusano, Marić, Tanigawa. We will need the statement of Proposition 3.1, with a slight modification, namely that condition (3.4) is replaced by the more general one:

$$\int_t^{\infty} h^{\beta}(s)\,\mathrm{d}s \le \frac{1}{\alpha-1} a^{\beta-1}(t)h(t), \quad t \ge t_0,$$

where $a(t)$ is a continuous nonincreasing function satisfying

$$0 < a(t) \le a < \frac{1}{\alpha}\left(\frac{\alpha-1}{\alpha}\right)^{\alpha-1}$$

for some constant a. The proof of such modified proposition is almost the same as that of the original one, and so it is omitted.

Theorem 3.5. *Suppose that the hypotheses of Proposition 3.1 with the above modification are satisfied. Let there exist a positive integer n such that*

$$\int^{\infty} a^{n(\beta-1)}(t)h^{\beta-1}(t)\,\mathrm{d}t < \infty \quad \textit{if} \quad 1 < \alpha \le 2, \tag{3.43}$$

$$\int^{\infty} a^{n(\beta-1)^2}(t)h^{\beta-1}(t)\,\mathrm{d}t < \infty \quad \textit{if} \quad \alpha > 2. \tag{3.44}$$

Then, for the solution (3.6) *of* (3.2)*, the following asymptotic formula holds for $t \to \infty$*

$$y(t) \sim B\exp\left\{\int_{t_0}^{t} \Phi^{-1}[v_{n-1}(s) + P(s)]\,\mathrm{d}s\right\}, \tag{3.45}$$

where B is a positive constant. Here the sequence $\{v_n(t)\}$ of successive approximations is defined by

$$v_0(t) = 0, \quad v_n(t) = (\alpha - 1)\int_t^\infty |v_{n-1}(s) + P(s)|^\beta \,\mathrm{d}s, \quad n = 1, 2, \ldots. \tag{3.46}$$

Proof. Let y be the solution given by (3.6). Recall that the function v used in (3.6) has been constructed as the fixed element in $C_h[t_0, \infty)$ of the contractive mapping $\mathcal{T}$ defined by (3.9). The standard proof of the contraction mapping principle shows that the sequence $\{v_n(t)\}$ defined by (3.46) converges to $v(t)$ uniformly on $[t_0, \infty)$. To see how fast $v_n(t)$ approaches $v(t)$ we proceed as follows. First, note that $|v_n(t)| \le (\alpha - 1)h(t)$, $t \ge t_0$, $n = 1, 2, \ldots$. By definition, we have

$$|v_1(t)| = (\alpha - 1)\int_t^\infty |P(s)|^\beta \,\mathrm{d}s \le (\alpha - 1)\int_t^\infty h^\beta(s)\,\mathrm{d}s \le a^{\beta-1}(t)h(t),$$

and

$$\begin{aligned}
|v_2(t) - v_1(t)| &\le (\alpha - 1)\int_t^\infty \Big||v_1(s) + P(s)|^\beta - |P(s)|^\beta\Big| \,\mathrm{d}s \\
&\le (\alpha - 1)\beta \int_t^\infty [\alpha h(s)]^\beta |v_1(s)| \,\mathrm{d}s \\
&\le \alpha^\beta \int_t^\infty a^{\beta-1}(s)h^\beta(s)\,\mathrm{d}s \le \alpha^\beta a^{\beta-1}(t)\int_t^\infty h^\beta(s)\,\mathrm{d}s \\
&\le (\beta - 1)\alpha^\beta a^{2(\beta-1)}(t)h(t) \le \gamma_\alpha^{\beta-1}\left(\frac{a(t)}{\gamma_\alpha}\right)^{2(\beta-1)} h(t)
\end{aligned}$$

for $t \ge t_0$, where $\gamma_\alpha = \frac{1}{\alpha}\left(\frac{\alpha-1}{\alpha}\right)^{\alpha-1}$. Assuming that

$$|v_n(t) - v_{n-1}(t)| \le \gamma_\alpha^\beta \left(\frac{a(t)}{\gamma_\alpha}\right)^{n(\beta-1)} h(t), \tag{3.47}$$

$t \ge t_0$, for some $n \in \mathbb{N}$, we compute

$$\begin{aligned}
|v_{n+1}(t) - v_n(t)| &\le (\alpha - 1)\int_t^\infty \Big||P(s) + v_n(s)|^\beta - |P(s) + v_{n-1}(s)|^\beta\Big| \,\mathrm{d}s \\
&\le (\alpha - 1)\beta \int_t^\infty [\alpha h(s)]^{\beta-1} |v_n(s) - v_{n-1}(s)| \,\mathrm{d}s \\
&= \alpha^\beta \int_t^\infty \gamma_\alpha^{\beta-1}\left(\frac{a(s)}{\gamma_\alpha}\right)^{n(\beta-1)} h^\beta(s)\,\mathrm{d}s \\
&\le \alpha^\beta \gamma_\alpha^{\beta-1}\left(\frac{a(t)}{\gamma_\alpha}\right)^{n(\beta-1)} \int_t^\infty h^\beta(s)\,\mathrm{d}s \\
&\le \alpha^\beta \gamma_\alpha^{\beta-1}\left(\frac{a(t)}{\gamma_\alpha}\right)^{n(\beta-1)} (\beta - 1)a^{\beta-1}(t)h(t) \\
&= \gamma_\alpha^{\beta-1}\left(\frac{a(t)}{\gamma_\alpha}\right)^{n(\beta-1)} h(t),
\end{aligned}$$

$t \geq t_0$, which establishes the validity of (3.47) for all integers $n \in \mathbb{N}$. Now we have

$$v(t) = v_{n-1}(t) + \sum_{k=n}^{\infty} [v_k(t) - v_{k-1}(t)],$$

from which, due to (3.47), it follows that

$$\begin{aligned} |v(t) - v_{n-1}(t)| &\leq \gamma_\alpha^{\beta-1} \left(\frac{a(t)}{\gamma_\alpha}\right)^{k(\beta-1)} h(t) \\ &\leq \gamma_\alpha^{\beta-1} \left(\frac{a(t)}{\gamma_\alpha}\right)^{n(\beta-1)} \sum_{k=0}^{\infty} \left(\frac{a}{\gamma_\alpha}\right)^k h(t) \\ &= \gamma_\alpha \left(\frac{a(t)}{\gamma_\alpha}\right)^{n(\beta-1)} \frac{\gamma_\alpha}{\gamma_\alpha - a} h(t) = K a^{n(\beta-1)}(t) h(t), \end{aligned} \tag{3.48}$$

where K is a constant depending only on α and n. Using (3.6) and (3.48), we obtain

$$\begin{aligned} & y(t) \left(\exp\left\{ \int_{t_0}^{t} \Phi^{-1}[P(s) + v_{n-1}(s)]\, \mathrm{d}s \right\} \right)^{-1} \\ &\qquad = \exp\left\{ \int_{t_0}^{t} \left(\Phi^{-1}[P(s) + v(s)] - \Phi^{-1}[P(s) + v_{n-1}(s)] \right) \mathrm{d}s \right\}. \end{aligned} \tag{3.49}$$

Let $1 < \alpha \leq 2$. Then, by the Mean Value Theorem and (3.48),

$$\begin{aligned} \left|\Phi^{-1}[P(t) + v(t)] - \Phi^{-1}[P(t) + v_{n-1}(t)]\right| &\leq (\beta - 1)[\alpha h(t)]^{\beta-2} |v(t) - v_{n-1}(t)| \\ &\leq N a^{n(\beta-1)}(t) h^{\beta-1}(t), \end{aligned} \tag{3.50}$$

$t \geq t_0$, where N is a constant depending on α and n. Let $\alpha > 2$. Then, using (3.48) and the inequality $|a^\lambda - b^\lambda| \leq 2|a - b|^\lambda$ holding for $\lambda \in (0, 1)$ and $a, b \in \mathbb{R}$, we see that

$$\begin{aligned} \left|\Phi^{-1}[P(t) + v(t)] - \Phi^{-1}[P(t) + v_{n-1}(t)]\right| &\leq 2|v(t) - v_{n-1}(t)|^{\beta-1} \\ &\leq M a^{n(\beta-1)^2}(t) h^{\beta-1}(t), \end{aligned} \tag{3.51}$$

$t \geq t_0$, where M is a constant depending on α and n. Combining (3.49) with (3.50) or (3.51) according as $1 < \alpha \leq 2$ or $\alpha > 2$, and using (3.43) or (3.44), we conclude that the right-hand side of (3.49) tends to a constant $B > 0$ as $t \to \infty$, which implies that $y(t)$ has the desired asymptotic behavior (3.45). □

Corollary 3.1. *Suppose that* (3.11) *holds and that the function* $a(t) = \varphi(t)$*, where* $\varphi(t)$ *is defined by* (3.12)*, satisfies*

$$\int^{\infty} \frac{a^{(n+1)(\beta-1)}(t)}{t}\, \mathrm{d}t < \infty \quad \textit{if} \quad 1 < \alpha \leq 2, \tag{3.52}$$

$$\int^{\infty} \frac{a^{(n+\alpha-1)(\beta-1)^2}}{t}\, \mathrm{d}t < \infty \quad \textit{if} \quad \alpha > 2. \tag{3.53}$$

Then the formula (3.45) *holds for the slowly varying solution* $y(t)$ *of* (3.2)*.*

Proof. The conclusion follows from the previous theorem combined with the observation that in this case

$$a^{n(\beta-1)}(t)h^{\beta-1}(t) = \frac{a^{(n+1)(\beta-1)}(t)}{t} \quad\text{and}\quad a^{n(\beta-1)^2}(t)h^{\beta-1}(t) = \frac{a^{(n+\alpha-1)(\beta-1)^2}(t)}{t}$$

according to whether $1 < \alpha \le 2$ or $\alpha > 2$. □

3.4 More general case

It is a natural question whether a generalization of (some of) the results from Section 3.2 to equation (3.1) is possible.

First idea which can come into mind is a transformation into equation of the form (3.2). Unfortunately, in contrast to the linear case we do not have a (linear) transformation of dependent variable $y = hu$ at disposal. Thus it is impossible to "kill" the coefficient r in general. Equation (3.1) can be transformed into an equation of the form (3.2) with preserving unboundedness and the form of the interval only when $\int_a^\infty r^{1-\beta}(s)\,\mathrm{d}s = \infty$. Indeed, we introduce new independent variable by $s = \xi(t) = \int_a^t r^{1-\beta}(s)\,\mathrm{d}s$ and new function $x(s) = y(t)$. Then (3.1) is transformed into

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(r(\xi^{-1}(s))\Phi(\xi'(\xi^{-1}(s)))\Phi\left(\frac{\mathrm{d}}{\mathrm{d}s}x\right)\right) + \frac{p(\xi^{-1}(s))}{\xi'(\xi^{-1}(s))}\Phi(x) = 0,$$

where ξ^{-1} is the inverse of ξ, or

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\Phi\left(\frac{\mathrm{d}}{\mathrm{d}s}x\right)\right) + \frac{p(\xi^{-1}(s))}{r^{1-\beta}(\xi^{-1}(s))}\Phi(x) = 0.$$

Jaroš, Kusano, and Tanigawa in [66] used a different approach, based on the same idea as for the linear equation in Theorems 2.8 and 2.9, namely utilizing the concept of generalized varying functions (see Definition 1.10) with the combination of the approach described in Section 3.2. The two (logical) cases are distinguished:

$$\int_a^\infty r^{1-\beta}(s)\,\mathrm{d}s = \infty \tag{3.54}$$

and

$$\int_a^\infty r^{1-\beta}(s)\,\mathrm{d}s < \infty. \tag{3.55}$$

We denote $R_\alpha(t) = \int_a^t r^{1-\beta}(s)\,\mathrm{d}s$ in case (3.54) and $\tilde{R}_\alpha(t) = \int_t^\infty r^{1-\beta}(s)\,\mathrm{d}s$ in case (3.55).

Let $\lambda_1 < \lambda_2$ denote the two real roots of the equation (3.13). A generalization of Theorem 3.1 and of Theorem 3.2 reads as follows; at the same time it can be viewed as a generalization of Theorem 2.8-(i).

Theorem 3.6. *Let* (3.54) *hold. Equation* (3.1) *is nonoscillatory and has two solutions* y_1, y_2 *such that* $y_i(t) \in \mathcal{NRV}_{R_\alpha}(\Phi^{-1}(\lambda_i))$, $i = 1, 2$, *if and only if*

$$\lim_{t\to\infty} R_\alpha^{\alpha-1}(t) \int_t^\infty p(s)\,\mathrm{d}s = A \in \left(-\infty, \frac{1}{\alpha}\left(\frac{\alpha-1}{\alpha}\right)^{\alpha-1}\right). \tag{3.56}$$

Proof. Since the proof uses essentially the same ideas as the proof of Theorem 3.1 and of Theorem 3.2, we mention only a few facts. Denote $\omega(t) = R_\alpha^{\alpha-1}(t) \int_t^\infty p(s)\,\mathrm{d}s - A$. The solutions y_i, when $\lambda_1 \neq 0$, are sought in the form

$$y_i(t) = \exp\left\{\int_{t_i}^t \Phi^{-1}\left(\frac{\lambda_i + \omega(s) + v_i(s)}{r(s)R_\alpha^{\alpha-1}(s)}\right)\mathrm{d}s\right\},$$

$i = 1, 2$. The function v_i is chosen in such a way that $w_i = (\lambda_i + \omega(t) + v_i)/R^{\alpha-1}$ satisfies the generalized Riccati equation

$$w' + p(t) + (\alpha - 1)r^{1-\beta}(t)|w|^\beta = 0. \tag{3.57}$$

To find the desired v_i, the contraction mapping theorem is used.

In the case when $\lambda_1 = 0$ (i.e., $A = 0$), the solution y_1 is sought in the form

$$y_1(t) = \exp\left\{\int_{t_1}^t \Phi^{-1}\left(\frac{R_\alpha^{\alpha-1}(s)\int_s^\infty p(\tau)\,\mathrm{d}\tau + w_1(s)}{r(s)R_\alpha^{\alpha-1}(s)}\right)\mathrm{d}s\right\},$$

where the differential equation for w_1 has the form

$$\left(\frac{w_1}{R_\alpha^{\alpha-1}(s)}\right)' + \frac{(\alpha-1)\left|R_\alpha^{\alpha-1}(t)\int_t^\infty p(\tau)\,\mathrm{d}\tau + w_1(s)\right|^\beta}{r^{\beta-1}(s)R_\alpha^\alpha(s)} = 0.$$

□

The next result is a generalization of Theorem 3.3 in case (3.54); at the same time it generalizes Theorem 2.8-(ii). We assume

$$\lim_{t\to\infty} R^{\alpha-1}(t)\int_t^\infty p(s)\,\mathrm{d}s = \frac{1}{\alpha}\left(\frac{\alpha-1}{\alpha}\right)^{\alpha-1}. \tag{3.58}$$

Equation (3.1) with p satisfying this condition can be regarded as a perturbation of the generalized Euler equation

$$(r(t)\Phi(y'))' + \frac{\bar{\gamma}}{r^{\beta-1}(t)R_\alpha^\alpha(t)}\Phi(y) = 0, \quad \bar{\gamma} = \left(\frac{\alpha-1}{\alpha}\right)^\alpha,$$

which is nonoscillatory. We show that (3.1) has a solution in $\mathcal{NRV}_{R_\alpha}$ provided the perturbation is small in some sense.

Theorem 3.7. *Suppose that* (3.54) *and* (3.58) *hold. Put*

$$\Upsilon(t) = R_\alpha^{\alpha-1}(t)\int_t^\infty p(s)\,\mathrm{d}s - \frac{1}{\alpha}\left(\frac{\alpha-1}{\alpha}\right)^{\alpha-1}$$

and suppose that

$$\int^\infty \frac{|\Upsilon(t)|}{r^{\beta-1}(t)R_\alpha(t)}\,\mathrm{d}t < \infty$$

and

$$\int^\infty \frac{\Psi(t)}{r^{\beta-1}(t)R_\alpha(t)}\,\mathrm{d}t < \infty, \text{ where } \Psi(t) = \int_t^\infty \frac{|\Upsilon(s)|}{r^{\beta-1}(s)R_\alpha(s)}\,\mathrm{d}s.$$

Then equation (3.1) *is nonoscillatory and has a solution* $y \in \mathcal{NRV}_{R_\alpha}((\alpha-1)/\alpha)$ *of the form* $y(t) = R_\alpha^{(\alpha-1)/\alpha}(t)L(t)$ *with* $L \in \mathcal{NSV}_{R_\alpha}$ *and* $\lim_{t\to\infty} L(t) = \ell \in (0,\infty)$.

Proof. Similarly as in the previous proof we omit details since it uses similar arguments as the proof of Theorem 3.3. Note just that we look for a solution of (3.1) expressed in the form

$$y(t) = \exp\left\{\int_{t_i}^t \Phi^{-1}\left(\frac{\bar{\gamma} + \Upsilon(s) + v(s)}{r(s)R_\alpha^{\alpha-1}(s)}\right)\mathrm{d}s\right\},$$

where v is desired to satisfy a certain Riccati type equation; the existence of v is proved by means of the contraction mapping theorem. □

We now turn to the case where r in (3.1) satisfies (3.55). We give a half-linear extension of Theorem 2.9-(i), which is at the same time a counterpart of Theorem 3.6.

Let $\sigma_1 < \sigma_2$ denote the two real roots of the equation $|\sigma|^\beta + \sigma + B = 0$.

Theorem 3.8. *Let* (3.55) *hold. Equation* (3.1) *is nonoscillatory and has two solutions* y_1, y_2 *such that* $y_i(t) \in \mathcal{NRV}_{1/\tilde{R}_\alpha}(\Phi^{-1}(\sigma_i))$, $i = 1, 2$, *if and only if*

$$\lim_{t\to\infty}\frac{1}{\tilde{R}_\alpha(t)}\int_t^\infty \tilde{R}_\alpha^\alpha(s)p(s)\,\mathrm{d}s = B \in \left(-\infty, \left(\frac{\alpha-1}{\alpha}\right)^\alpha\right).$$

Proof. The proof uses similar ideas as the proof of Theorem 3.7. Therefore we only mention a few facts. The solutions y_i, $i = 1, 2$, when $\sigma_2 \neq 0$, are sought in the form

$$y_i(t) = \exp\left\{\int_{t_i}^t \Phi^{-1}\left(\frac{\sigma_i + \omega(s) + v_i(s)}{r(s)\tilde{R}_\alpha^{\alpha-1}(s)}\right)\mathrm{d}s\right\},$$

$i = 1, 2$, where $\omega(t) = \frac{1}{\tilde{R}_\alpha(t)}\int_t^\infty \tilde{R}_\alpha^\alpha(s)p(s)\,\mathrm{d}s - B$. The function v_i is chosen in such a way that $w_i = (\sigma_i + \omega(t) + v_i)/\tilde{R}^{\alpha-1}$ satisfies the generalized Riccati equation (3.57). To find the desired v_i, the contraction mapping theorem is used again. The case when $\sigma_2 = 0$ (i.e., $B = 0$) is examined separately; the differential equation for w_2 to be solved has the form

$$w_2' + \frac{\alpha - 1}{r^{\beta-1}(t)\tilde{R}_\alpha(t)} w_2 + + \frac{\alpha - 1}{r^{\beta-1}(t)\tilde{R}_\alpha(t)} [|\omega(t) + w_2|\beta + \beta\omega(t)] = 0.$$

□

The counterpart of Theorem 3.7 which is at the same time a generalization of Theorem 2.9-(ii) reads as follows. Equation (3.1) can be seen again as a perturbation of certain Euler type equation, this time of the form

$$(r(t)\Phi(y'))' + \frac{\bar{\gamma}}{r^{\beta-1}(t)\tilde{R}_\alpha^\alpha(t)} \Phi(y) = 0.$$

Theorem 3.9. *Let* (3.55) *hold. Assume that*

$$\lim_{t\to\infty} \frac{1}{\tilde{R}_\alpha(t)} \int_t^\infty \tilde{R}_\alpha^\alpha(s) p(s)\,\mathrm{d}s = \left(\frac{\alpha - 1}{\alpha}\right)^\alpha.$$

Put

$$\Upsilon(t) = \frac{1}{\tilde{R}_\alpha(t)} \int_t^\infty \tilde{R}_\alpha^\alpha(s) p(s)\,\mathrm{d}s - \left(\frac{\alpha - 1}{\alpha}\right)^\alpha$$

and suppose that

$$\int^\infty \frac{|\Upsilon(t)|}{r^{\beta-1}(t)\tilde{R}_\alpha(t)}\,\mathrm{d}t < \infty$$

and

$$\int^\infty \frac{\Psi(t)}{r^{\beta-1}(t)\tilde{R}_\alpha(t)}\,\mathrm{d}t < \infty, \quad \textit{where} \quad \Psi(t) = \int_t^\infty \frac{|\Upsilon(s)|}{r^{\beta-1}(s)\tilde{R}_\alpha(s)}\,\mathrm{d}s.$$

Then equation (3.1) *is nonoscillatory and has a solution* $y \in \mathcal{NRV}_{1/\tilde{R}_\alpha}(-(\alpha-1)/\alpha)$ *of the form* $y(t) = \tilde{R}_\alpha^{(\alpha-1)/\alpha}(t)L(t)$ *with* $L \in \mathcal{NSV}_{1/\tilde{R}_\alpha}$ *and* $\lim_{t\to\infty} L(t) = \ell \in (0, \infty)$.

Proof. We omit details again. Note only that we seek a solution of (3.1) expressed in the form

$$y(t) = \exp\left\{\int_{t_i}^t \Phi^{-1}\left(\frac{\Upsilon(s) - \bar{\gamma} + v(s)}{r(s)\tilde{R}_\alpha^{\alpha-1}(s)}\right) \mathrm{d}s\right\},$$

where v is desired to satisfy a certain Riccati type equation; the existence of v is proved by means of the contraction mapping theorem. □

3.5 Asymptotic formulas for nonoscillatory solutions of conditionally oscillatory half-linear equations

In this section which is based on the paper [137] by Pátíková we investigate asymptotic properties of nonoscillatory solutions of a special conditionally oscillatory half-linear second order differential equation, which was constructed in [29] as a perturbation of equation (3.1).

Throughout this section we suppose that (3.1) is nonoscillatory. Let $d(t)$ be a positive continuous function, we say that the equation

$$(r(t)\Phi(x'))' + [p(t) + \mu d(t)]\Phi(x) = 0 \tag{3.59}$$

is *conditionally oscillatory* if there exists a constant $\mu_0 > 0$ such that (3.59) is oscillatory for $\mu > \mu_0$ and nonoscillatory for $\mu < \mu_0$. Let $h(t)$ be a positive solution of nonoscillatory equation (3.1) such that $h'(t) \neq 0$ on some interval of the form $[T_0, \infty)$ and denote

$$R(t) := r(t)h^2(t)|h'(t)|^{\alpha-2}, \quad G(t) := r(t)h(t)\Phi(h'(t)). \tag{3.60}$$

Under the assumptions

$$\int^{\infty} \frac{dt}{R(t)} = \infty, \quad \liminf_{t\to\infty} |G(t)| > 0,$$

the authors of [29] constructed a conditionally oscillatory equation seen as a perturbation of (3.1) in the form

$$(r(t)\Phi(x'))' + \left[p(t) + \frac{\mu}{h^{\alpha}(t)R(t)\left(\int^t R^{-1}(s)\,\mathrm{d}s\right)^2}\right]\Phi(x) = 0. \tag{3.61}$$

The critical oscillation constant of this equation is $\mu_0 = \frac{1}{2\beta}$, where β is the conjugate number to α. In [29] it is also shown that (3.61) has for this constant $\mu = \mu_0$ a solution with the asymptotic formula

$$x(t) = h(t)\left(\int^t R^{-1}(s)\,ds\right)^{\frac{1}{\alpha}}\left(1 + O\left(\left(\int^t R^{-1}(s)\,\mathrm{d}s\right)^{-1}\right)\right) \quad \text{as} \quad t \to \infty. \tag{3.62}$$

The aim of this section is to present more precise asymptotic formulas in terms of slowly and regularly varying functions in the case where the constant μ is less than or equal to $\frac{1}{2\beta}$.

The "perturbation approach", when the considered equation is regarded as a perturbation of another half-linear equation, has been also used in [135]. There, the asymptotics of nonoscillatory solutions of

$$(\Phi(x'))' + \frac{\gamma_\alpha}{t^\alpha}\Phi(x) + \tilde{p}(t)\Phi(x) = 0, \tag{3.63}$$

where $\gamma_\alpha = \left(\frac{\alpha-1}{\alpha}\right)^{\alpha}$, was established under the assumption

$$\lim_{t\to\infty} \log t \int_t^{\infty} \tilde{p}(s)s^{\alpha-1}\,\mathrm{d}s \in \left(-\infty, \frac{1}{2}\left(\frac{\alpha-1}{\alpha}\right)^{\alpha-1}\right].$$

Equation (3.63) has been seen as a perturbation of the half-linear Euler type equation

$$(\Phi(x'))' + \frac{\gamma_\alpha}{t^\alpha}\Phi(x) = 0 \tag{3.64}$$

with the critical constant γ_α.

It is well-known (see, e.g., [28]), that similarly as in the linear oscillation theory, the nonoscillation of equation (3.1) is equivalent to the solvability of a Riccati type equation. In particular, if x is an eventually positive or negative solution of the nonoscillatory equation (3.1) on some interval of the form $[T_0, \infty)$, then $w(t) = r(t)\Phi\left(\frac{x'}{x}\right)$ solves the Riccati type equation

$$w' + p(t) + (\alpha - 1)r^{1-\beta}(t)|w|^\beta = 0. \tag{3.65}$$

Conversely, having a solution $w(t)$ of (3.65) for $t \in [T_0, \infty)$, the corresponding solution of (3.1) can be expressed as

$$x(t) = C \exp\left\{\int^t r^{1-\beta}(s)\Phi^{-1}(w)\,\mathrm{d}s\right\},$$

where Φ^{-1} is the inverse function of Φ and C is a constant.

Using the concept of perturbations it appears useful to deal with the so called modified (or generalized) Riccati equation. Let h be a positive solution of (3.1) and $w_h(t) = r(t)\Phi\left(\frac{h'}{h}\right)$ be the corresponding solution of the Riccati equation (3.65). Let us consider another nonoscillatory equation

$$(r(t)\Phi(x'))' + P(t)\Phi(x) = 0 \tag{3.66}$$

and let $w(t)$ be a solution of the Riccati equation associated with (3.66). Then $v(t) = (w(t) - w_h(t))h^\alpha(t)$ solves the modified Riccati equation

$$v' + (P(t) - p(t))h^\alpha + \alpha r^{1-\beta}h^\alpha \tilde{P}(\Phi^{-1}(w_h), w) = 0, \tag{3.67}$$

where

$$\tilde{P}(u, v) := \frac{|u|^\alpha}{\alpha} - uv + \frac{|v|^\beta}{\beta} \geq 0,$$

with the equality $\tilde{P}(u, v) = 0$ if and only if $v = \Phi(u)$. We deal with this equation in a slightly different, but still equivalent, form

$$v' + (P(t) - p(t))h^\alpha + (\alpha - 1)r^{1-\beta}h^{-\beta}|G|^\beta F\left(\frac{v}{G}\right) = 0, \tag{3.68}$$

where $G(t)$ is defined by (3.60) and

$$F(u) = |u + 1|^\beta - \beta u - 1. \tag{3.69}$$

Next we apply the perturbation principle combined with the (modified) Riccati technique to get asymptotical results for (3.61) with $\mu < \frac{1}{2\beta}$ and $\mu = \frac{1}{2\beta}$.

Theorem 3.10. *Suppose that* (3.1) *is nonoscillatory and possesses a positive solution* $h(t)$ *such that* $h'(t) \neq 0$ *for large* t *and let*

$$\int^{\infty} \frac{\mathrm{d}t}{R(t)} = \infty, \tag{3.70}$$

and

$$\liminf_{t\to\infty} |G(t)| > 0. \tag{3.71}$$

If $\mu < \frac{1}{2\beta}$, *then the conditionally oscillatory equation* (3.61) *has a pair of solutions given by the asymptotic formula*

$$x_i = h(t)\left(\int^t R^{-1}(s)\,\mathrm{d}s\right)^{(\beta-1)\lambda_i} L_i(t),$$

where λ_i *are zeros of the quadratic equation*

$$\frac{\beta}{2}\lambda^2 - \lambda + \mu = 0 \tag{3.72}$$

and $L_i(t)$ *are generalized normalized slowly varying functions of the form*

$$L_i(t) = \exp\left\{\int^t \frac{\varepsilon_i(s)}{R(s)\int^s R^{-1}(\tau)\,d\tau}\,ds\right\}$$

and $\varepsilon_i(t) \to 0$ *for* $t \to \infty$.

Proof. We are looking for solutions of the modified Riccati equation associated with (3.61), which reads as

$$v'(t) + \frac{\mu}{R(t)\left(\int^t R^{-1}(s)\,\mathrm{d}s\right)^2} + (\alpha-1)r^{1-\beta}(t)h^{-\beta}(t)|G(t)|^{\beta}F\left(\frac{v(t)}{G(t)}\right) = 0, \tag{3.73}$$

where G is defined by (3.60) and F by (3.69).

Assumptions (3.70) and (3.71) imply the convergence of the integral

$$\int^{\infty} r^{1-\beta}(t)h^{-\beta}(t)|G(t)|^{\beta}F\left(\frac{v(t)}{G(t)}\right)\mathrm{d}t,$$

from which it follows (see [29]) that $v(t) \to 0$ and $\frac{v(t)}{G(t)} \to 0$ as $t \to \infty$.

Let $C_0[T,\infty)$ be the set of all continuous functions on the interval $[T,\infty)$ (T will be specified later) which converge to zero a $t \to \infty$ and let us consider a set of functions

$$V = \{\omega \in C_0[T,\infty) : |\omega(t)| < \varepsilon, t \geq T\},$$

where $\varepsilon > 0$ is so small that

$$\frac{1}{\sqrt{1-2q\mu}}(q+1)\,\varepsilon < \frac{1}{2}. \tag{3.74}$$

Let us also observe that the inequality

$$\frac{1}{\sqrt{1-2q\mu}}\left(\frac{q}{2}+1\right)\varepsilon \le 1 \tag{3.75}$$

is implied. Denote the roots of the quadratic equation (3.72) for $\mu < \frac{1}{2q}$ as

$$\lambda_1 = \frac{1-\sqrt{1-2\beta\mu}}{\beta}, \quad \lambda_2 = \frac{1+\sqrt{1-2\beta\mu}}{\beta}.$$

We assume that two solutions of the modified Riccati equation (3.73) are in the form

$$v_i(z,t) := \frac{\lambda_i + z(t)}{\int^t R^{-1}(s)\,\mathrm{d}s},$$

for $t \in [T,\infty)$ and $z \in V$, $i = 1,2$. Substituting this function and its derivative into (3.73), we have

$$z'(t) + \frac{\mu - \lambda_i - z(t)}{R(t)\int^t R^{-1}(s)\,\mathrm{d}s} + (\alpha-1)r^{1-\beta}(t)h^{-\beta}(t)|G(t)|^{\beta}F\left(\frac{v_i(z,t)}{G(t)}\right)\int^t R^{-1}(s)\,\mathrm{d}s = 0$$

which can be rewritten as

$$z'(t) \quad + \quad \frac{(-1+\lambda_i\beta)z(t)}{R(t)\int^t R^{-1}(s)\,\mathrm{d}s} + \frac{1}{R(t)\int^t R^{-1}(s)\,\mathrm{d}s}E_i(z,t) = 0, \tag{3.76}$$

where

$$E_i(z,t) := \mu - \lambda_i - \lambda_i\beta z(t) + (\alpha-1)\left(\int^t R^{-1}(s)\,\mathrm{d}s\right)^2 G^2(t)F\left(\frac{v_i(z,t)}{G(t)}\right).$$

This means that looking for solutions v_i of the modified Riccati equation (3.73) is equivalent to looking for solutions z_i of the equation (3.76). In the next we shall show that two solutions of (3.76) can be found through the Banach fixed-point theorem used onto suitable integral operators.

First, let us turn our attention to the behavior of the function $F(u)$, which plays an important role in estimating of certain useful expressions. Studying the behavior of $F(u)$ and $F'(u)$ for u in a neighborhood of 0, we have

$$\begin{aligned} F(u) &= \frac{F''(0)}{2}u^2 + \frac{F'''(\zeta)}{6}u^3 \\ &= \frac{\beta(\beta-1)}{2}u^2 + \frac{\beta(\beta-1)(\beta-2)}{6}|1+\zeta|^{\beta-3}\operatorname{sgn}(1+\zeta)u^3, \end{aligned}$$

where ζ is between 0 and u. For $|u| < \frac{1}{2}$ and hence also $|\zeta| < \frac{1}{2}$ there exists a positive constant M_q such that

$$\left|\frac{\beta(\beta-2)}{6}|1+\zeta|^{\beta-3}\operatorname{sgn}(1+\zeta)\right| \leq M_\beta \quad \text{for} \quad \beta > 1.$$

Therefore

$$\left|F(u) - \frac{\beta(\beta-1)}{2}u^2\right| \leq (\beta-1)M_\beta|u|^3. \tag{3.77}$$

Similarly,

$$\begin{aligned} F'(u) &= F''(0)u + \frac{F'''(\zeta')}{2}u^2 \\ &= \beta(\beta-1)u + \frac{\beta(\beta-1)(\beta-2)}{2}|1+\zeta'|^{\beta-3}\operatorname{sgn}(1+\zeta')u^2, \end{aligned}$$

where ζ' is between 0 and u. Again, considering $|\zeta'| < \frac{1}{2}$ we have

$$\left|F'(u) - \beta(\beta-1)u\right| \leq 3(\beta-1)M_\beta|u|^2. \tag{3.78}$$

Now, let us denote $J(t) := \int^t R^{-1}(s)\,ds$, then the estimate of the function $E_i(z,t)$ for $z \in V$, reads as

$$\begin{aligned} |E_i(z,t)| &= \Bigg|\mu - \lambda_i - \lambda_i\beta z(t) + \frac{\beta}{2}J^2(t)v_i^2(z,t) + (\alpha-1)J^2(t)G^2(t)F\left(\frac{v_i(z,t)}{G(t)}\right) \\ &\quad - \frac{\beta}{2}J^2(t)v_i^2(z,t)\Bigg| \leq |\mu - \lambda_i - \lambda_i\beta z(t) + \frac{\beta}{2}(\lambda_i + z(t))^2| \\ &\quad + \left|(\alpha-1)J^2(t)G^2(t)\left[F\left(\frac{v_i(z,t)}{G(t)}\right) - \frac{\beta(\beta-1)}{2}\left(\frac{v_i(z,t)}{G(t)}\right)^2\right]\right| \\ &\leq \frac{\beta}{2}|z(t)|^2 + \frac{M_\beta|\lambda_i + z(t)|^3}{|G(t)||J(t)|} \leq \frac{\beta}{2}|z(t)|^2 + \frac{KM_\beta|\lambda_i + z(t)|^3}{|J(t)|}, \end{aligned}$$

where (3.77) was used for $u = \frac{v_i}{G}$ and $K := \sup_{t \geq T} \frac{1}{|G(t)|}$ is a finite constant for T sufficiently large because of (3.71). According to (3.70) there exists T_1 such that the last term in the previous inequality is less than ε^2 and therefore

$$|E_i(z,t)| \leq \frac{q}{2}\varepsilon^2 + \varepsilon^2 \leq \varepsilon^2\left(\frac{q}{2} + 1\right) \tag{3.79}$$

for $t \geq T_1$. Furthermore, for $z_1, z_2 \in V$ we have

$$\begin{aligned} |E_i(z_1,t) - E_i(z_2,t)| &= \Bigg| - \lambda_i\beta(z_1 - z_2) \\ &\quad + (p-1)J^2(t)G^2(t)\left[F\left(\frac{v_i(z_1,t)}{G(t)}\right) - F\left(\frac{v_i(z_2,t)}{G(t)}\right)\right]\Bigg|, \end{aligned}$$

which, by the mean value theorem with a suitable $z(t) \in V$, becomes

$$\begin{aligned}
&= \Bigg| -\lambda_i q(z_1 - z_2) + \beta J(t) v_i(z,t)(z_1 - z_2) \\
&\qquad + (\alpha - 1) J(t) G(t) F' \left(\frac{v_i(z,t)}{G(t)} \right) (z_1 - z_2) - \beta J(t) v_i(z,t)(z_1 - z_2) \Bigg| \\
&\le \Bigg| -\lambda_i \beta + \beta J(t) v_i(z,t) \\
&\qquad + (\alpha - 1) J(t) G(t) \left(F' \left(\frac{v_i(z,t)}{G(t)} \right) - \beta(\beta - 1) \frac{v_i(z,t)}{G(t)} \right) \Bigg| \cdot \|z_1 - z_2\| \\
&\le \left(\beta |z(t)| + \left| \frac{3KM_q(\lambda_i + z(t))^2}{J(t)} \right| \right) \cdot \|z_1 - z_2\|,
\end{aligned}$$

where (3.78) was used. Similarly as in the previous estimate, there exists T_2 such that the middle term in the last row of the inequality is less than ε and hence

$$|E_i(z_1, t) - E_i(z_2, t)| \le \varepsilon(\beta + 1) \cdot \|z_1 - z_2\| \tag{3.80}$$

for $t \in [T_2, \infty)$.

Now, let us consider the pair of functions

$$r_i(t) := \exp \left\{ \int^t \frac{-1 + \lambda_i \beta}{R(s) \int^s R^{-1}(\tau)\, \mathrm{d}\tau} \, \mathrm{d}s \right\}, \; i = 1, 2.$$

Then equation (3.76) is equivalent to

$$(r_i(t) z(t))' + r_i(t) \frac{1}{R(t) \int^t R^{-1}(s)\, \mathrm{d}s} E_i(z, t) = 0. \tag{3.81}$$

For $i = 1$, we have the function

$$r_1(t) = \exp \left\{ \int^t \frac{-1 + \lambda_1 \beta}{R(s) \int^s R^{-1}(\tau)\, \mathrm{d}\tau} \, \mathrm{d}s \right\} = \exp \left\{ \int^t \frac{-\sqrt{1 - 2\beta\mu}}{R(s) \int^s R^{-1}(\tau)\, \mathrm{d}\tau} \, \mathrm{d}s \right\}$$

and it is easy to see that $r_1(t) \to 0$ for $t \to \infty$.

Finally, let us define the integral operator F_1 on the set of functions V by

$$(F_1 z)(t) = \frac{1}{r_1(t)} \int_t^\infty \frac{r_1(s)}{R(s) \int^s R^{-1}(\tau)\, \mathrm{d}\tau} E_1(z, s)\, \mathrm{d}s.$$

We observe that

$$\int_t^\infty \frac{r_1(s)}{R(s) \int^s R^{-1}(\tau)\, \mathrm{d}\tau} \, \mathrm{d}s = \frac{r_1(t)}{\sqrt{1 - 2\beta\mu}}.$$

Taking $T = \max\{T_1, T_2\}$, by (3.79) and (3.74) we have

$$\begin{aligned} |(F_1 z)(t)| &\le \frac{1}{r_1(t)} \int_t^\infty \frac{r_1(s)}{R(s) \int^s R^{-1}(\tau)\, \mathrm{d}\tau} |E_1(z,s)|\, \mathrm{d}s \\ &\le \frac{1}{\sqrt{1-2\beta\mu}} \left(\frac{\beta}{2} + 1\right) \varepsilon^2 \le \varepsilon, \end{aligned}$$

which means that F_1 maps the set V into itself, and by (3.80) and (3.75) we see that

$$\begin{aligned} |(F_1 z_1)(t) - (F_1 z_2)(t)| \le \frac{1}{r_1(t)} \int_t^\infty \frac{r_1(s)}{R(s) \int^s R^{-1}(\tau)\, d\tau} |E_1(z_1,s) - E_1(z_2,s)|\, \mathrm{d}s \\ \le \|z_1 - z_2\| \frac{1}{\sqrt{1-2\beta\mu}} \varepsilon(\beta+1) < \frac{1}{2} \|z_1 - z_2\|, \end{aligned}$$

which implies that F_1 is a contraction. Using the Banach fixed-point theorem we can find a function $z_1(t)$, that satisfies $z_1 = F_1 z_1$. That means that $z_1(t)$ is a solution of (3.81) and also of (3.76) and $v_1(t) = \frac{\lambda_1 + z_1(t)}{\int^t R^{-1}\, ds}$ is a solution of (3.73). For $i = 2$ we have

$$r_2(t) = \exp\left\{ \int^t \frac{-1 + \lambda_2 \beta}{R(s) \int^s R^{-1}(\tau)\, \mathrm{d}\tau}\, \mathrm{d}s \right\} = \exp\left\{ \int^t \frac{\sqrt{1-2q\mu}}{R(s) \int^s R^{-1}(\tau)\, \mathrm{d}\tau}\, \mathrm{d}s \right\}$$

and we define an integral operator F_2 by

$$(F_2 z)(t) = -\frac{1}{r_2(t)} \int^t \frac{r_2(s)}{R(s) \int^s R^{-1}(\tau)\, \mathrm{d}\tau} E_2(z,s)\, \mathrm{d}s.$$

Since

$$\int^t \frac{r_2(s)}{R(s) \int^s R^{-1}(\tau)\, \mathrm{d}\tau}\, \mathrm{d}s = \frac{r_2(t) - c}{\sqrt{1-2\beta\mu}},$$

where c is a positive suitable constant, the inequality

$$\frac{1}{r_2(t)} \int^t \frac{r_2(s)}{R(s) \int^s R^{-1}(\tau)\, \mathrm{d}\tau}\, \mathrm{d}s \le \frac{1}{\sqrt{1-2\beta\mu}}$$

holds for t sufficiently large, as $r_2(t) \to \infty$ for $t \to \infty$. Taking $T = \max\{T_1, T_2\}$, the estimates for the operator F_2 are the same as in the previous case and we can find a fixed point $z_2(t)$ satisfying $F_2 z_2 = z_2$. Thus $z_2(t)$ solves (3.81) and $v_2(t) = \frac{\lambda_2 + z_2(t)}{\int^t R^{-1}\, ds}$ solves the modified Riccati equation (3.73).

Expressing the solutions of the "standard" Riccati equation for (3.61) corresponding to the solutions $v_i(z_i, t)$ of the modified Riccati equation, we have

$$\begin{aligned} w_i(t) &= h^{-\alpha}(t) v_i(z_i, t) + w_h(t) = w_h(t) \left(1 + \frac{v_i(z_i,t)}{h^\alpha(t) w_h(t)}\right) \\ &= w_h(t) \left(1 + \frac{\lambda_i + z_i(t)}{h^\alpha(t) w_h(t) \int^t R^{-1}\, \mathrm{d}s}\right) = w_h(t) \left(1 + \frac{\lambda_i + z_i(t)}{G(t) \int^t R^{-1}(s)\, \mathrm{d}s}\right). \end{aligned}$$

Since solutions of (3.61) are given by the formula

$$x(t) = \exp\left\{\int^t r^{1-\beta}(s)\Phi^{-1}(w)\,\mathrm{d}s\right\},$$

we need to express

$$\begin{aligned}
r^{1-\beta}(t)\Phi^{-1}(w_i) &= \frac{h'(t)}{h(t)}\left(1+\frac{\lambda_i+z_i(t)}{G(t)\int^t R^{-1}(s)\,\mathrm{d}s}\right)^{\beta-1} \\
&= \frac{h'(t)}{h(t)}\left(1+(\beta-1)\frac{\lambda_i+z_i(t)}{G(t)\int^t R^{-1}(s)\,\mathrm{d}s}+o\left(\frac{\lambda_i+z_i(t)}{G(t)\int^t R^{-1}(s)\,\mathrm{d}s}\right)\right) \\
&= \frac{h'(t)}{h(t)}+\frac{(\beta-1)\lambda_i}{R(t)\int^t R^{-1}(s)\,\mathrm{d}s} \\
&\quad +\frac{(\beta-1)z_i(t)}{R(t)\int^t R^{-1}(s)\,\mathrm{d}s}+o\left(\frac{\lambda_i+z_i(t)}{R(t)\int^t R^{-1}(s)\,\mathrm{d}s}\right).
\end{aligned}$$

Because

$$\begin{aligned}
o\left(\frac{\lambda_i+z_i(t)}{R(t)\int^t R^{-1}(s)\,\mathrm{d}s}\right) &= \frac{R(t)\int^t R^{-1}(s)\,\mathrm{d}s\, o\left(\frac{\lambda_i+z_i(t)}{R(t)\int^t R^{-1}(s)\,\mathrm{d}s}\right)}{R(t)\int^t R^{-1}(s)\,\mathrm{d}s} \\
&= \frac{o(\lambda_i+z_i(t))}{R(t)\int^t R^{-1}(s)\,\mathrm{d}s}
\end{aligned}$$

holds for large t, the pair of solutions of (3.61) for $i = 1, 2$ is in the form

$$\begin{aligned}
x_i(t) = \exp\Bigg\{&\ln h(t) + \ln\left(\int^t R^{-1}(s)\,\mathrm{d}s\right)^{(\beta-1)\lambda_i} \\
&+\int^t \frac{(\beta-1)z_i(s)+o(\lambda_i+z_i(s))}{R(s)\int^s R^{-1}(\tau)\,\mathrm{d}\tau}\Bigg\}.
\end{aligned}$$

As $z_i \in V$ and hence $z_i(t) \to 0$ for $t \to \infty$, the statement of the theorem holds for $\varepsilon_i(t) = (\beta-1)z_i(t) + o(\lambda_i + z_i(t))$. □

Now let us present the asymptotic formula in case $\mu = \frac{1}{2\beta}$, which gives an improved version of (3.62).

Theorem 3.11. *Let the assumptions of the previous theorem be satisfied and let $\mu = \frac{1}{2q}$. Then equation* (3.61) *has a solution of the form*

$$x(t) = h(t)\left(\int^t R^{-1}(s)\,\mathrm{d}s\right)^{\frac{1}{p}} L(t), \tag{3.82}$$

where $L(t)$ is a generalized normalized slowly varying function of the form

$$L(t) = \exp\left\{\int^t \frac{\varepsilon(s)}{R(s)\int^s R^{-1}(\tau)\,\mathrm{d}\tau}\,\mathrm{d}s\right\}$$

and $\varepsilon(t) \to 0$ for $t \to \infty$.

Proof. For $\mu = \frac{1}{2\beta}$ the quadratic equation (3.72) has the double root $\lambda = \frac{1}{\beta}$. We assume the solution of modified Riccati equation to be in the form (for $z \in V$)

$$v(z,t) = \frac{\frac{1}{\beta} + z(t)}{\int^t R^{-1}(s)\,\mathrm{d}s},$$

which gives, after substituting into the modified Riccati equation (3.73) for $\mu = \frac{1}{2\beta}$,

$$z'(t) + \frac{-z(t) - \frac{1}{2\beta}}{R(t)\int^t R^{-1}(s)\,\mathrm{d}s} + (\alpha-1)r^{1-\beta}(t)h^{-\beta}(t)|G(t)|^{\beta}F\left(\frac{v(z,t)}{G(t)}\right)\int^t R^{-1}(s)\,\mathrm{d}s = 0.$$

Let us denote

$$E(z,t) = -z(t) - \frac{1}{2\beta} + (\alpha-1)\left(\int^t R^{-1}(s)\,\mathrm{d}s\right)^2 G^2(t)F\left(\frac{v(z,t)}{G(t)}\right)$$

and let us consider an integral operator F_3

$$(F_3 z)(t) = \int_t^{\infty} \frac{1}{R(s)\int^s R^{-1}(\tau)\,\mathrm{d}\tau} E(z,s)\,\mathrm{d}s$$

on a set of continuous functions

$$V = \{\omega \in C_0[T,\infty) : |\omega(t)| < \varepsilon, t \geq T\},$$

where T and ε are to be established similarly as in the proof of the previous theorem. Then the solution of modified Riccati equation and also the solution of the studied equation can be found in almost the same manner as for the previous statement. □

Remark 3.1. If $r(t) \equiv 1, p(t) = \gamma_\alpha t^{-\alpha}$ and $h(t) = t^{\frac{\alpha-1}{\alpha}}$ then the conditionally oscillatory equation (3.61) with $\mu = \frac{1}{2\beta}$, seen as a perturbation of the Euler equation (3.64), becomes the Euler-Weber (or alternatively Riemann-Weber) half-linear differential equation

$$(\Phi(x'))' + \left[\frac{\gamma_\alpha}{t^\alpha} + \frac{\mu_\alpha}{t^\alpha \ln^2 t}\right]\Phi(x) = 0$$

with the so-called critical coefficient $\mu_p = \frac{1}{2}\left(\frac{\alpha-1}{\alpha}\right)^{\alpha-1}$. The asymptotic formula (3.82) then reduces to the formula given in [135, Theorem 2].

Remark 3.2. For the Euler-Weber half-linear equation also the asymptotic formula for its second linearly independent solution is known (see [136]). An open question remains whether the second linearly independent solution of (3.61) with $\mu = \frac{1}{2\beta}$ could be found in a similar form

$$x_2(t) = h(t)\left(\int^t R^{-1}(s)\,\mathrm{d}s\right)^{\frac{1}{\alpha}} \left(\ln\left(\int^t R^{-1}(s)\,\mathrm{d}s\right)\right)^{\frac{2}{\alpha}} L_2(t),$$

where

$$L_2(t) = \exp\left\{\int^t \frac{\varepsilon_2(s)}{R(s)\int^s R^{-1}(\tau)\,\mathrm{d}\tau \ln\left(\int^s R^{-1}(\tau)\,\mathrm{d}\tau\right)}\,\mathrm{d}s\right\}$$

and $\varepsilon_2(t) \to 0$ as $t \to \infty$.

3.6 De Haan type solutions

3.6.1 Solutions in the class Γ

We consider the equations

$$(r(t)\Phi(y'))' = p(t)\Phi(y), \tag{3.83}$$

and

$$(\Phi(y'))' = p(t)\Phi(y), \tag{3.84}$$

where r, p are positive continuous functions on $[a, \infty)$. In this subsection we deal with solutions of (3.84) which are shown to be in the class Γ, but $\mathcal{RV}$ solutions are also examined. The results are taken from [147] by Řehák. Later we briefly discuss possible extensions (based on Řehák, Taddei [150]).

Because of the sign condition on p, equation (3.83) is nonoscillatory by the Sturm type comparison theorem; it suffices to compare (3.83) with $(r(t)\Phi(y'))' = 0$. For a (nonoscillatory) solution y of (3.83) it holds, $y(t)y'(t) > 0$ or $y(t)y'(t) < 0$ eventually. Without loss of generality we may work just with positive solutions. Denote

$$\mathbb{M}^+ = \{y : y \text{ is a solution of (3.83)}, y(t) > 0, y'(t) > 0 \text{ for large } t\}.$$

Under our sign condition, the class $\mathbb{M}^+$ is nonempty. Moreover,

$$\begin{cases} \forall (a_0, a_1) \in (0,\infty)\times(0,\infty)\ \forall t_0 \text{ sufficiently} \\ \text{large } \exists y \in \mathbb{M}^+ : y(t_0) = a_0, y'(t_0) = a_1. \end{cases} \tag{3.85}$$

Further, denote $\mathbb{M}^+_B = \{y \in \mathbb{M}^+ : \lim_{t\to\infty} y(t) < \infty\}$ and $\mathbb{M}^+_\infty = \{y \in \mathbb{M}^+ : \lim_{t\to\infty} y(t) = \infty\}$. Clearly, $\mathbb{M}^+ = \mathbb{M}^+_B \cup \mathbb{M}^+_\infty$. Set

$$J = \int_a^\infty r^{1-\beta}(t)\left(\int_a^t p(s)\,\mathrm{d}s\right)^{\beta-1}\mathrm{d}t \quad\text{and}\quad J_r = \int_a^\infty r^{1-\beta}(s)\,\mathrm{d}s.$$

There hold $J<\infty \Rightarrow J_r<\infty$ and $\mathbb{M}^+_B \neq \emptyset \Leftrightarrow J<\infty$. Hence,

$$J_r=\infty \Rightarrow (\emptyset \neq)\, \mathbb{M}^+ = \mathbb{M}^+_\infty. \tag{3.86}$$

In the special case of equation (3.84), that is $r(t)=1$, the discussed classes will be denoted as $\mathbb{M}^+(1)$ and $\mathbb{M}^+_\infty(1)$. For basic properties (classification and existence) of nonoscillatory solutions see e.g. [28, Section 4.1].

Theorem 3.12. *If $p^{-\frac{1}{\alpha}} \in \mathcal{BSV}$, then $\emptyset \neq \mathbb{M}^+(1) = \mathbb{M}^+_\infty(1) \subseteq \Gamma\left(\left(\frac{\alpha-1}{p}\right)^{\frac{1}{\alpha}}\right)$.*

Proof. In the first part of the proof, we assume that $p \in C^1$. From the assumption of the theorem, in view of properties of $\mathcal{BSV}$ functions, we have $(p^{-\frac{1}{\alpha}})'(t) = -p'(t)/(\alpha p^{\frac{\alpha+1}{\alpha}}(t)) \to 0$ as $t \to \infty$. We know that $\mathbb{M}^+(1) \neq \emptyset$. Take $y \in \mathbb{M}^+(1)$. Then $y \in \mathbb{M}^+_\infty(1)$. Alternatively we can start with $y(t) \to \infty$, and then we find $y \in \mathbb{M}^+(1) = \mathbb{M}^+_\infty(1)$. If we set $w = \frac{f\Phi(y')}{\Phi(y)}$, where f is C^1 nonzero function, then w satisfies the generalized Riccati equation

$$w' - \frac{f'(t)}{f(t)} w - f(t)p(t) + (\alpha-1) f^{1-\beta}(t)|w|^\beta = 0$$

for large t, see [28, Section 2.2.2]. Put $f = p^{-\frac{1}{\beta}}$. Then $w>0$ and the Riccati like equation becomes

$$\frac{w'}{p^{\frac{1}{\alpha}}(t)} = 1-(\alpha-1)w\left(\frac{p'(t)}{\alpha p^{\frac{\alpha+1}{\alpha}}(t)} + w^{\beta-1}\right) \tag{3.87}$$

for large t. We want to prove that $\lim_{t\to\infty} w(t) = (\alpha-1)^{-\frac{1}{\beta}}$. We distinguish three cases according to the eventual sign of w'. First, let $w'(t)>0$ for large t. Then $\lim_{t\to\infty} w(t) = A \in (0,\infty) \cup \{\infty\}$. If $A=\infty$, then from (3.87) and $p'(t)/(p^{\frac{\alpha+1}{\alpha}}(t)) \to 0$, we have $\lim_{t\to\infty} w'(t)p^{-\frac{1}{\alpha}}(t) = -\infty$, a contradiction with $w'(t)>0$. Let $A \in (0,\infty)$. Then from (3.87) and $p'(t)/(p^{\frac{\alpha+1}{\alpha}}(t)) \to 0$, we have $\lim_{t\to\infty} w'(t)p^{-\frac{1}{\alpha}}(t) = 1-(\alpha-1)A^\beta$. If $A \neq (\alpha-1)^{-\frac{1}{\beta}}$, then $w'(t) \sim (1-(\alpha-1)A^\beta)p^{\frac{1}{\alpha}}(t)$ as $t\to\infty$, hence

$$m_1 \int_{t_0}^t p^{\frac{1}{\alpha}}(s)\,\mathrm{d}s \le w(t)-w(t_0) \le M_1 \int_{t_0}^t p^{\frac{1}{\alpha}}(s)\,\mathrm{d}s, \tag{3.88}$$

$t \ge t_0$, t_0 being large, for some $0<m_1<M_1<\infty$. From $w = p^{-\frac{1}{\beta}}(y'/y)^{\alpha-1}$, we have $w^{\beta-1} = p^{-\frac{1}{\alpha}} y'/y$. Since $w^{\beta-1}(t) \le M$, $t \ge t_0$, for some $M \in (0,\infty)$, we get $p^{\frac{1}{\alpha}}(t) \ge \frac{y'(t)}{y(t)M}$, $t \ge t_0$. Hence,

$$\int_{t_0}^t p^{\frac{1}{\alpha}}(s)\,\mathrm{d}s \ge \frac{1}{M}\int_{t_0}^t \frac{y'(s)}{y(s)}\,\mathrm{d}s = \frac{1}{M}\ln\frac{y(t)}{y(t_0)},$$

$t \ge t_0$, which implies $\int_{t_0}^t p^{\frac{1}{\alpha}}(s)\,\mathrm{d}s \to \infty$ as $t\to\infty$. In view of (3.88), we obtain $\lim_{t\to\infty} w(t) = \infty$, a contradiction with $A<\infty$. Thus, $\lim_{t\to\infty} w(t) = (\alpha-1)^{-\frac{1}{\beta}}$. If

$w'(t) < 0$ for large t, then we use similar arguments; in particular, we distinguish the cases when $\lim_{t\to\infty} w(t) = 0$ and $\lim_{t\to\infty} w(t) > 0$, and reach a contradiction in undesirable cases. Finally we assume $w'(t_n) = 0$ for a sequence $\{t_n\}$ with $t_n \to \infty$ as $n \to \infty$. We take here zeroes of w' being consecutive; the finite cluster point cannot exist. From (3.87), we have

$$w^\beta(t_n) - \frac{1}{\alpha - 1} = -\frac{p'(t_n)}{\alpha} p^{-\frac{\alpha+1}{\alpha}}(t_n) w(t_n). \tag{3.89}$$

Hence, $w(t)$ hits the positive real root of the equation

$$|\lambda|^\beta - \frac{1}{\alpha - 1} = -\frac{p'(t_n)}{\alpha} p^{-\frac{\alpha+1}{\alpha}}(t_n)\lambda$$

at each $t = t_n$. The positive root indeed exists; it is sufficient to realize that the left hand side of the equation represents a parabola like curve symmetric w.r.t. the vertical axis, and with the negative minimum, while the right hand side is a line which goes through the origin. The function w is monotone between zeroes of w', thus $w(t_n) \le w(t) \le w(t_{n+1})$ or $w(t_{n+1}) \le w(t) \le w(t_n)$, $t_n \le t \le t_{n+1}$. Thanks to $p'(t)/(p^{\frac{\alpha+1}{\alpha}}(t)) \to 0$ as $t \to \infty$, from (3.89) we get $\lim_{n\to\infty} w(t_n) = (\alpha - 1)^{-\frac{1}{\beta}}$, and so $\lim_{t\to\infty} w(t) = (\alpha - 1)^{-\frac{1}{\beta}}$. Thus altogether we have $\lim_{t\to\infty} p^{-\frac{1}{\beta}}(t)(y'(t)/y(t))^{\alpha-1} = (\alpha - 1)^{-\frac{1}{\beta}}$, which implies

$$\left(\frac{y(t)}{y'(t)}\right)^\alpha \sim \left(\frac{\alpha - 1}{p(t)}\right)^{\frac{1}{\beta}\cdot\frac{\alpha}{\alpha-1}} = \frac{\alpha - 1}{p(t)} \quad \text{as } t \to \infty. \tag{3.90}$$

Further, because of the identity $(\Phi(y'))' = (\alpha - 1)y''|y'|^{\alpha-2}$, from (2.1) we get

$$\frac{y''y}{y'^2} = \frac{y''y(y')^{\alpha-2}}{y'^\alpha} = \frac{(\Phi(y'))'y}{(\alpha - 1)y'^\alpha} = \frac{p\Phi(y)y}{(\alpha - 1)y'^\alpha} = \frac{p}{\alpha - 1}\left(\frac{y}{y'}\right)^\alpha. \tag{3.91}$$

From (3.90) and (3.91), we obtain

$$\frac{y''(t)y(t)}{y'^2(t)} = \frac{p(t)}{\alpha - 1}\left(\frac{y(t)}{y'(t)}\right)^\alpha \sim \frac{p(t)}{\alpha - 1} \cdot \frac{\alpha - 1}{p(t)} = 1$$

as $t \to \infty$. Hence, $y \in \Gamma(h)$, where $h = y/y'$. From (3.90), $h(t) \sim \left(\frac{\alpha-1}{p(t)}\right)^{\frac{1}{\alpha}}$ as $t \to \infty$, thus $y \in \Gamma\left(\left(\frac{\alpha-1}{p}\right)^{\frac{1}{\alpha}}\right)$.

Now we drop the assumption of differentiability of p. Since $p^{-\frac{1}{\alpha}} \in \mathcal{BSV}$, there exists $\hat{p} \in C^1$ such that $\hat{p}(t) \sim p(t)$ and $(\hat{p}^{-\frac{1}{\alpha}})'(t) \to 0$ as $t \to \infty$. For every $\varepsilon \in (0, 1)$ there exists a (large) t_0 such that $(1 - \varepsilon)\hat{p}(t) \le p(t) \le (1 + \varepsilon)\hat{p}(t)$, $t \ge t_0$. Consider the two auxiliary equations $(\Phi(u'))' = (1 + \varepsilon)\hat{p}(t)\Phi(u)$ resp. $(\Phi(v'))' = (1 - \varepsilon)\hat{p}(t)\Phi(v)$. Take their solutions u resp. v which are in the relevant $\mathbb{M}^+_\infty(1)$ type classes, and satisfy $u(t_0) = u_0 > 0$, $u'(t_0) = u_1 > 0$ resp. $v(t_0) = v_0 > 0$, $v'(t_0) = v_1 > 0$, with

u_0, u_1, v_0, v_1 being determined more precisely later. Such solutions indeed exist by (3.85) and (3.86). Set $w_u = \Phi(u'/u)$, resp. $w_v = \Phi(v'/v)$, resp. $w_y = \Phi(y'/y)$. These (eventually positive) functions then satisfy the generalized Riccati equations $w_u' = (1+\varepsilon)\hat{p}(t) - (\alpha-1)|w_u|^\beta$, resp. $w_v' = (1-\varepsilon)\hat{p}(t) - (\alpha-1)|w_v|^\beta$, resp. $w_y' = p(t) - (\alpha-1)|w_y|^\beta$, $t \ge t_0$. Because of arbitrariness of v_0, v_1 (see (3.85)) we may take v in the $\mathbb{M}_\infty^+(1)$ type class in such a way that $w_v(t_0) \le w_y(t_0)$. From the standard result on differential inequalities ([53, Chapter III, Section 4]), we have $w_v(t) \le w_y(t)$, $t \ge t_0$. Hence,

$$\left(\frac{v'(t)}{v(t)}\right)^{\alpha-1} p^{-\frac{1}{\beta}}(t) \le \left(\frac{y'(t)}{y(t)}\right)^{\alpha-1} p^{-\frac{1}{\beta}}(t),$$

$t \ge t_0$. As in the first part of the proof, we get

$$\left(\frac{v'(t)}{v(t)}\right)^{\alpha-1} \hat{p}^{-\frac{1}{\beta}}(t) \sim \left(\frac{1-\varepsilon}{\alpha-1}\right)^{\frac{1}{\beta}},$$

which yields, in view of $p(t) \sim \hat{p}(t)$,

$$\left(\frac{v'(t)}{v(t)}\right)^{\alpha-1} p^{-\frac{1}{\beta}}(t) \sim \left(\frac{1-\varepsilon}{\alpha-1}\right)^{\frac{1}{\beta}}$$

as $t \to \infty$. Thus we get

$$\liminf_{t\to\infty} \left(\frac{y'(t)}{y(t)}\right)^{\alpha-1} p^{-\frac{1}{\beta}}(t) \ge \left(\frac{1-\varepsilon}{\alpha-1}\right)^{\frac{1}{\beta}}.$$

Analogously, examining w_u, we get

$$\limsup_{t\to\infty} \left(\frac{y'(t)}{y(t)}\right)^{\alpha-1} p^{-\frac{1}{\beta}}(t) \le \left(\frac{1+\varepsilon}{\alpha-1}\right)^{\frac{1}{\beta}}.$$

Since $\varepsilon \in (0,1)$ was arbitrary, we obtain

$$\lim_{t\to\infty} (y'(t)/y(t))^{\alpha-1} p^{-\frac{1}{\beta}}(t) = (\alpha-1)^{-\frac{1}{\beta}}.$$

The rest of the proof is the same as in the previous part. $\square$

Results in the spirit of Theorem 3.12 for linear equation are presented in Subsection 2.6.1 Recall that $\Gamma \subset \mathcal{RPV}(\infty)$. Thus we should mention also Theorem 2.1, where the condition

$$t \int_t^{\lambda t} p(s)\,\mathrm{d}s \to \infty$$

as $t \to \infty$ is proved to be necessary and sufficient for decreasing and increasing solutions of (2.2) to be rapidly varying; the proof is presented just for decreasing solutions. Concerning an extension of this result to half-linear equation (3.84), the author has not found any paper on this topic, except of [121], where the discrete counterpart is considered and only decreasing solutions are examined. As for differential equations, there is only a mention made by prof. Kusano at CDDE conference (Brno, 2000) about an allegedly existing statement which extends Marić's theorem for the case of decreasing solutions to (3.84), with emphasizing that the case of increasing solutions of (3.84) is an open problem (both, the existence of an $\mathcal{RPV}$ increasing solution as well as the fact that all increasing solutions are $\mathcal{RPV}$). Theorem 3.12, in fact, offers a condition guaranteeing that any eventually positive increasing solution (which indeed exists) is rapidly varying and, moreover, its asymptotic behavior is more specified. Observe that this sufficient condition, for a differentiable p having the form $(p^{-\frac{1}{\alpha}})'(t) \to 0$ as $t \to \infty$, implies the one of Marić type modified for (3.84), namely

$$t^{\alpha-1} \int_t^{\lambda t} p(s)\,\mathrm{d}s \to \infty.$$

Theorem 3.12 can be seen also as an extension of the classical result by Hartman-Wintner [54, Paragraph 24]. Finally, consider the special case when $p(t) \to C \in (0, \infty)$ as $t \to \infty$. Then, from the proof of Theorem 3.12 we see that $y \in \mathbb{M}^+$ satisfies $\lim_{t\to\infty} y'(t)/y(t) = ((\alpha-1)/C)^{-\frac{1}{\alpha}}$. Thus, from this point of view, the solution y can be understood as of Poincaré-Perron type. Such a behavior, along with numerous refinements, is studied quite extensively for linear equations or systems also in the present; as one of the pioneering works can be regarded the classical paper [140]. So as a by-product of our results, we offer — to some extent — a half-linear extension of this result in the second order case.

Now we somehow modify the above ideas which will lead to regularly varying behavior. Let p be differentiable. In the previous considerations we assumed that $\lim_{t\to\infty}(p^{-\frac{1}{\alpha}})'(t) = 0$. Now assume that the limit is nonzero, i.e., $\lim_{t\to\infty} p'(t)/p^{-\frac{\alpha+1}{\alpha}} = C \neq 0$. Denote by $\hat{\varrho}$ the positive root of

$$|\varrho|^\beta + \frac{C}{\alpha}\varrho - \frac{1}{\alpha-1} = 0.$$

It is easy to see that the root indeed exists and $\hat{\varrho} \neq (\alpha-1)^{-\frac{1}{\beta}}$. Take $y \in \mathbb{M}^+(1)$. Similar arguments as in the first part of the proof of Theorem 3.12 yield

$$\left(\frac{y'(t)}{y(t)}\right)^{\alpha-1} p^{-\frac{1}{\beta}}(t) \sim \hat{\varrho}$$

as $t \to \infty$. Using this relation along with (3.91), we get

$$\lim_{t\to\infty} \frac{y''(t)y(t)}{y'^2(t)} = \frac{1}{(\alpha-1)\hat{\varrho}^\beta} \neq 1.$$

We (naturally) assume that $C < 0$. Then $\hat{\varrho} > (\alpha-1)^{-\beta}$, and so

$$\lim_{t\to\infty} \frac{y''(t)y(t)}{y'^2(t)} = \sigma < 1,$$

where $\sigma = 1/((\alpha-1)\hat\varrho^\beta)$. Hence, $y \in \mathcal{RV}(1/(1-\sigma))$, or $y \in \mathcal{RV}(-\alpha\hat\varrho^{\beta-1}/C)$. Thanks to convexity of solutions to (2.1), we get normalized regular variation. Thus we have proved the following theorem.

Theorem 3.13. *Let p be differentiable and $\lim_{t\to\infty} p'(t)/p^{-\frac{\alpha+1}{\alpha}}(t) = C < 0$. Then*

$$\emptyset \neq \mathbb{M}^+(1) = \mathbb{M}^+_\infty(1) \subseteq \mathcal{NRV}(-\alpha\hat\varrho^{\beta-1}/C),$$

where $\hat\varrho$ is the positive root of

$$|\varrho|^\beta + \frac{C}{\alpha}\varrho - \frac{1}{\alpha-1} = 0.$$

From the results of Section 3.2 it follows that (3.84) possesses solutions y_i with $y_i \in \mathcal{RV}(\Phi^{-1}(\lambda_i))$, $i = 1, 2$, where $\lambda_1 < 0 < \lambda_2$ are the roots of $|\lambda|^\beta - \lambda - A = 0$ if and only if $\lim_{t\to\infty} t^{\alpha-1}\int_t^\infty p(s)\,\mathrm{d}s = A(< 0)$. Observe that $p'(t)p^{-\frac{\alpha+1}{\alpha}}(t) \sim C(< 0)$ implies $t^{\alpha-1}\int_t^\infty p(s)\,\mathrm{d}s \sim (-\alpha/C)^\alpha/(\alpha-1)$ as $t \to \infty$. Further, it is easy to see that $\lambda_2 = (-\alpha/C)^{\alpha-1}\hat\varrho$, and so — as expected — the indices of regular variation of increasing solutions in both the results coincide. Note that the integral condition from Section 3.2 is more general than the condition in the previous theorem. On the other hand, the fixed point approach used in Section 3.2 guarantees the existence of at least one positive increasing $\mathcal{RV}$ solution, while the result in this section says that all positive increasing solutions are regularly varying.

Now we show a connection with generalized regular variation. First observe, that if a positive function $f \in C^1$ satisfies $\tau(t)f'(t)/f(t) \sim \vartheta \in \mathbb{R}$ as $t \to \infty$, where τ is positive continuous with $\int_a^\infty (1/\tau(s))\,\mathrm{d}s = \infty$, then $f \in \mathcal{NRV}_\omega(\vartheta)$, where $\omega(t) = \exp\left\{\int_a^t 1/\tau(s)\,\mathrm{d}s\right\}$. This fact easily follows from the representation theorem. Assume now $p^{-\frac{1}{\alpha}} \in \mathcal{BSV}$, and take $y \in \mathbb{M}^+(1)$. From the proof of Theorem 3.12, we have $y'(t)/y(t) \sim (p(t)/(\alpha-1))^{\frac{1}{\alpha}}$ as $t \to \infty$. Set $\tau(t) = p^{-\frac{1}{\alpha}}(t)$. Then $\ln y(t) \sim N\int_{t_0}^t (1/\tau(s))\,\mathrm{d}s$ as $t \to \infty$, for some $N \in (0, \infty)$, which implies $\int_{t_0}^\infty (1/\tau(s))\,\mathrm{d}s = \infty$. If we set $\vartheta = (\alpha-1)^{-\frac{1}{\alpha}}$, then y satisfies $\tau(t)y'(t)/y(t) \sim \vartheta$ as $t \to \infty$ and now it is easy to see that

$$y \in \mathcal{NRV}_\omega\left((\alpha-1)^{-\frac{1}{\alpha}}\right), \quad \text{where } \omega(t) = \exp\left\{\int_a^t p^{\frac{1}{\alpha}}(s)\,\mathrm{d}s\right\}.$$

The statement of Theorem 3.12 can therefore be reformulated in terms of generalized $\mathcal{RV}$ functions. In particular, we get

$$\mathbb{M}^+(1) \subseteq \mathcal{NRV}_\omega\left((\alpha-1)^{-\frac{1}{\alpha}}\right)$$

with the above defined ω.

Consider now more general equation (3.83), where the coefficient r satisfies $\int_a^\infty r^{1-\beta}(s)\,\mathrm{d}s = \infty$. Set $R(t) = \int_a^t r^{1-\beta}(s)\,\mathrm{d}s$ and denote by R^{-1} its inversion. Let us introduce new independent variable $s = \xi(t)$ and new function $u(s) = y(\xi^{-1}(s)) = y(t)$, where $\xi \in C^1$, $\xi(t) > 0, \xi'(t) > 0$, $t \in [a, \infty)$, and $\xi(t) \to \infty$ as $t \to \infty$. Then $\frac{\mathrm{d}}{\mathrm{d}t} = \xi'(\xi^{-1}(s))\frac{\mathrm{d}}{\mathrm{d}s}$, and (3.83) is transformed into

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(r(\xi^{-1}(s))\Phi(\xi'(\xi^{-1}(s)))\Phi\left(\frac{\mathrm{d}u}{\mathrm{d}s}\right)\right) = \frac{p(\xi^{-1}(s))}{\xi'(\xi^{-1}(s))}\Phi(u). \tag{3.92}$$

Set $\xi(t) = R(t)$. Then $\xi'(\xi^{-1}(s)) = r^{1-\beta}(R^{-1}(s))$ and (3.92) becomes

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\Phi\left(\frac{\mathrm{d}u}{\mathrm{d}s}\right)\right) = \tilde{p}(s)\Phi(u), \tag{3.93}$$

where $\tilde{p}(s) = \frac{p(R^{-1}(s))}{r^{1-\beta}(R^{-1}(s))}$. Take a solution y of (3.83) such that $y \in \mathbb{M}^+$. Then $y \in \mathbb{M}^+_\infty$, and — after the transformation — the corresponding solution u of (3.93) satisfies $u(s) \to \infty$ as $s \to \infty$ with $\frac{\mathrm{d}}{\mathrm{d}s}u(s) > 0$ eventually. If we now assume $\tilde{p}^{-\frac{1}{\alpha}} \in \mathcal{BSV}$, then as in the proof of Theorem 4.14 we get

$$\frac{\frac{\mathrm{d}}{\mathrm{d}s}u(s)}{u(s)} \sim \left(\frac{\tilde{p}(s)}{\alpha - 1}\right)^{\frac{1}{\alpha}}$$

as $s \to \infty$. Since $\frac{\mathrm{d}}{\mathrm{d}s} = r^{\beta-1}(t)\frac{\mathrm{d}}{\mathrm{d}t}$, the last asymptotic relation yields

$$\frac{r^{\beta-1}(t)y'(t)}{y(t)} \sim \left(\frac{r^{\beta-1}(t)p(t)}{\alpha - 1}\right)^{\frac{1}{\alpha}}$$

as $t \to \infty$, which yields $y'(t)/y(t) \sim 1/Q(t)$ as $t \to \infty$, where $Q(t) = [p(t)/((\alpha - 1)r(t))]^{-\frac{1}{\alpha}}$. Take $\varepsilon > 0$ and t_0 such that

$$\frac{1-\varepsilon}{Q(t)} \le \frac{y'(t)}{y(t)} \le \frac{1+\varepsilon}{Q(t)},$$

$t \ge t_0$. Assume $Q \in \mathcal{BSV}$. Then $Q \in \mathcal{SN}$, because of continuity. Let $\lambda > 0$ and take the integral between t and $t + \lambda Q(t)$, for which by the local uniform convergence the following relation holds,

$$\int_t^{t+\lambda Q(t)} \frac{1}{Q(v)}\,\mathrm{d}v = \int_0^\lambda \frac{Q(t)}{Q(t + zQ(t))}\,\mathrm{d}z \to \lambda$$

as $t \to \infty$. Thus we get

$$(1-\varepsilon)\lambda \le \liminf_{t\to\infty} \ln\frac{y(t+\lambda Q(t))}{y(t)} \le \limsup_{t\to\infty} \ln\frac{y(t+\lambda Q(t))}{y(t)} \le (1+\varepsilon)\lambda.$$

Letting $\varepsilon \to 0$ we come to $y \in \Gamma(Q)$. The above considerations are made under the assumptions $\tilde{p} \in \mathcal{BSV}$ and $Q \in \mathcal{BSV}$. If we assume p, r differentiable, then — after some rearrangement in the former equality — we get

$$\frac{\mathrm{d}}{\mathrm{d}s}\tilde{p}^{-\frac{1}{\alpha}}(s) = \left(-\frac{1}{\alpha}\left(\frac{r}{p}\right)^{\frac{\alpha+1}{\alpha}}\frac{p'r - pr'}{r^2} + (1-\beta)\left(\frac{r}{p}\right)^{\frac{1}{\alpha}}\frac{r'}{r}\right) \circ R^{-1}(s)$$

and

$$(\alpha-1)^{-\frac{1}{\alpha}}Q'(t)=\left(-\frac{1}{\alpha}\left(\frac{r}{p}\right)^{\frac{\alpha+1}{\alpha}}\frac{p'r-pr'}{r^2}\right)(t).$$

Thus we see that in the case of differentiable coefficients it is sufficient to require the expressions on the right-hand sides to tend to zero as s resp. t tends to infinity. So we have obtained the following result, which extends Theorem 3.12.

Theorem 3.14. *Let $\int_a^\infty r^{1-\beta}(s)\,\mathrm{d}s=\infty$. Denote $R(t)=\int_a^t r^{1-\beta}(s)\,\mathrm{d}s$. Then any of the two sets of conditions*

(i) $\left(\frac{p}{r^{1-\beta}}\right)^{-\frac{1}{\alpha}}\circ R^{-1}\in\mathcal{BSV}$, $\left(\frac{p}{r}\right)^{-\frac{1}{\alpha}}\in\mathcal{BSV}$,

or

(ii) *p,r are differentiable,* $\left(\left(\frac{p}{r}\right)^{-\frac{1}{\alpha}}\right)'(t)\to 0$ *and* $\left(\left(\frac{p}{r}\right)^{-\frac{1}{\alpha}}\frac{r'}{r}\right)(t)\to 0$ *as* $t\to\infty$

guarantees

$$\emptyset\neq\mathbb{M}^+=\mathbb{M}^+_\infty\subseteq\Gamma\left(\left(\frac{(\alpha-1)r}{p}\right)^{\frac{1}{\alpha}}\right).$$

A closer examination of the above observations reveals that the part of the proof of Theorem 3.12 starting with (3.90) can alternatively be shown via a different method, namely the one which was used to obtain Theorem 3.14.

The asymptotic conditions in the second part of Theorem 3.14 can be relaxed to $\left(\left(\frac{p}{r}\right)^{-\frac{1}{\alpha}}\right)'(t)\to 0$ and $\left(\left(\frac{p}{r^{1-\beta}}\right)^{-\frac{1}{\alpha}}\right)'(t)\cdot r^{\beta-1}(t)\to 0$ as $t\to\infty$.

The results of this subsection were further extended in [150] by Řehák and Taddei. Equation (3.83) is examined there directly (i.e., not via a transformation) no matter which of the conditions $\int^\infty r^{1-\beta}(s)\,\mathrm{d}s=\infty$ and $\int^\infty r^{1-\beta}(s)\,\mathrm{d}s<\infty$ happens. Moreover, not only increasing solutions but also decreasing solutions are considered.

3.6.2 Solutions in the class Π

In the paper [150] by Řehák and Taddei, half-linear extensions of the Geluk type results (see e.g. Theorem 2.11) were obtained. The paper is in preparation. We give — for illustration — only one statement without proof. By $\mathbb{M}^-$ we mean the set of all eventually positive solutions of (3.83).

Theorem 3.15. *Let $p\in\mathcal{RV}(\delta)$ and $r\in\mathcal{RV}(\delta+\alpha)$ with $\delta<-1$. If $L_p(t)/L_r(t)\to 0$ as $t\to\infty$, where L_p and L_r are $\mathcal{SV}$ components of p and r, respectively, then $\mathbb{M}^-\subset\mathcal{NSV}$. If $y\in\mathbb{M}^-$, then $-y\in\Pi(-ty'(t))$. Moreover,* *(i) If* $\int_a^\infty \left(\frac{sp(s)}{r(s)}\right)^{\frac{1}{\alpha-1}} \mathrm{d}s = \infty$, *then*

$$y(t) = \exp\left\{-\int_a^t \left(\frac{sp(s)}{-(\delta+1)r(s)}\right)^{\frac{1}{\alpha-1}} (1+o(1))\,\mathrm{d}s\right\}$$

and $y(t) \to 0$ *as* $t \to \infty$.

(ii) If $\int_a^\infty \left(\frac{sp(s)}{r(s)}\right)^{\frac{1}{\alpha-1}} \mathrm{d}s < \infty$, *then*

$$y(t) = y(\infty)\exp\left\{\int_t^\infty \left(\frac{sp(s)}{-(\delta+1)r(s)}\right)^{\frac{1}{\alpha-1}} (1+o(1))\,\mathrm{d}s\right\}$$

and $y(t) \to y(\infty) \in (0,\infty)$ *as* $t \to \infty$.

Note that if we want to deal with $\mathcal{SV}$ solutions in the complementary case $\delta > -1$, we must look for them in the class $\mathbb{M}^+$ (positive increasing solutions).

3.7 Half-linear differential equations having regularly varying solutions

Kusano and Marić in [94] deals with the question whether for any distinct real constants ϑ_1 and ϑ_2 there exists a differential equation of the form (3.1) which possesses a pair of solutions $y_i \in \mathcal{RV}(\vartheta_i)$, $i = 1,2$. The function ω which appears in the theorem is assumed to satisfy conditions from the definition of regularly varying functions with respect to ω, see Definition 1.10.

For any $|\vartheta_1| \neq |\vartheta_2|$ define

$$M(\vartheta_1,\vartheta_2) = \frac{|\vartheta_1|^\beta - |\vartheta_2|^\beta}{\vartheta_1 - \vartheta_2}, \quad N(\vartheta_1,\vartheta_2) = \frac{\vartheta_2|\vartheta_1|^\beta - \vartheta_1|\vartheta_2|^\beta}{\vartheta_1 - \vartheta_2}$$

and observe that

$$M(\vartheta_1,\vartheta_2) \lesseqgtr 0 \Leftrightarrow \vartheta_1 + \vartheta_2 \lesseqgtr 0.$$

Theorem 3.16. *Let* ϑ_1 *and* ϑ_2 *be any given real constants such that* $|\vartheta_1| \neq |\vartheta_2|$.

(i) Suppose that r *satisfies*

$$r^{1-\beta}(t) \sim K\omega^{M(\vartheta_1,\vartheta_2)-1}(t)\omega'(t) \tag{3.94}$$

as $t \to \infty$ *for some positive constant* K. *Let* $M(\vartheta_1,\vartheta_2) > 0$ *and* p *be conditionally integrable on* $[a,\infty)$. *Then equation* (3.1) *possesses a pair of solutions* $y_i \in \mathcal{NRV}_\omega(\Phi^{-1}(\vartheta_i))$, $i = 1,2$, *if and only if*

$$\lim_{t\to\infty} K^{\alpha-1}\omega^{(\alpha-1)M(\vartheta_1,\vartheta_2)}(t)\int_t^\infty p(s)\,\mathrm{d}s = \frac{N(\vartheta_1,\vartheta_2)}{M(\vartheta_1,\vartheta_2)}. \tag{3.95}$$

(ii) Suppose that r *satisfies*

$$r^{1-\beta}(t) = K\omega^{M(\vartheta_1,\vartheta_2)-1}(t)\omega'(t)$$

for some positive constant K. Let $M(\vartheta_1,\vartheta_2)<0$ *and* $\omega^{\alpha}M(\vartheta_1,\vartheta_2)p$ *be conditionally integrable on* $[a,\infty)$*. Then equation* (3.1) *possesses a pair of solutions* $y_i\in\mathcal{NRV}_{\omega}(\Phi^{-1}(\vartheta_i))$*,* $i=1,2$*, if and only if*

$$\lim_{t\to\infty}K^{\alpha-1}\omega^{|M(\vartheta_1,\vartheta_2)|}(t)\int_t^{\infty}\omega^{\alpha M(\vartheta_1,\vartheta_2)}p(s)\,\mathrm{d}s=\frac{(\alpha-1)N(\vartheta_1,\vartheta_2)}{|M(\vartheta_1,\vartheta_2)|}.$$

Proof. (i) Suppose $M(\vartheta_1,\vartheta_2)>0$. For arbitrary real numbers ϑ_1,ϑ_2 choose A in equation (3.13) as

$$A=\frac{N(\vartheta_1,\vartheta_2)}{M^{\alpha}(\vartheta_1,\vartheta_2)}.$$

Then, its only two real roots are

$$\lambda_i=\frac{\vartheta_i}{M^{\alpha-1}(\vartheta_1,\vartheta_2)},$$

$i=1,2$. Further, choose function r in equation (3.1) such that for some constant $K>0$,

$$r^{1-\beta}(t)\sim K\omega^{M(\vartheta_1,\vartheta_2)-1}(t)\omega'(t)$$

as $t\to\infty$. Whence due to $R_{\alpha}(t)=\int_0^t r^{1-\beta}(s)\,\mathrm{d}s$ and (3.94), condition (3.56) becomes (3.95) and an application of Theorem 3.6 asserts that equation (3.1) has two solutions

$$y_i\in\mathcal{NRV}_{R_{\alpha}}\left(\frac{\Phi^{-1}(\vartheta_i)}{M(\vartheta_1,\vartheta_2)}\right),$$

$i=1,2$. By applying the basic properties of generalized $\mathcal{RV}$ functions,

$$\mathcal{RV}_{R_{\alpha}}\left(\frac{\Phi^{-1}(\vartheta_i)}{M(\vartheta_1,\vartheta_2)}\right)=\mathcal{RV}_{\omega M(\vartheta_1,\vartheta_2)}\left(\frac{\Phi^{-1}(\vartheta_i)}{M(\vartheta_1,\vartheta_2)}\right)=\mathcal{RV}_{\omega}(\Phi^{-1}(\vartheta_i)),$$

$i=1,2$, whence we conclude that the solutions y_i, $i=1,2$ of (3.1) belong to $\mathcal{NRV}_{\omega}(\Phi^{-1}(\vartheta_i))$, $i=1,2$, as desired.

(ii) The proof of this part is similar to that of (i); this time we utilize Theorem 3.8. □

Chapter 4

Emden-Fowler type equations and systems

4.1 Introductory and historical remarks

One of typical examples of the objects studied in this chapter is the second order *Emden-Fowler type* (or the *generalized Emden-Fowler* or the *quasilinear*) equation

$$(r(t)\Phi_\alpha(y'))' + p(t)\Phi_\gamma(y) = 0, \tag{4.1}$$

$\Phi_\lambda(u) := |u|^\lambda$, where $\alpha, \gamma \in (0,\infty)$ and $r > 0, p$ are continuous on $[a,\infty)$. If $\alpha = \gamma$, then (4.1) reduces to the half-linear equation of the form (3.2). Note that the power in Φ_λ is shifted by one in comparison with the power in Φ which was used in the previous chapter. But since here we start with zero, while formerly it was with one, the both nonlinearities are practically the same. We decided for these conventions because they are very usual in the literature when studying half-linear and quasi-linear differential equations.

First we give few historical remarks. In the study of stellar structure, the *Lane-Emden* equation

$$y''(t) + \frac{2}{t}y'(t) + y^\gamma(t) = 0 \tag{4.2}$$

was considered. This equation was proposed by Lane [102] and studied in detail by Emden [31]. Fowler [41, 42] considered a generalization of this equation, called *Emden-Fowler* equation

$$ty'' + Cy'(t) + Dt^\sigma y^\gamma = 0.$$

Note that (4.2) has the self-adjoint form $(t^2y')' + t^2y^\gamma = 0$. By the change of variable $t = 1/\xi$, (4.2) becomes $\frac{\mathrm{d}^2y}{\mathrm{d}\xi^2} + \frac{y^\gamma}{\xi^4} = 0$, and by the change of variable $y = \eta/t$, $\frac{\mathrm{d}^2\eta}{\mathrm{d}t^2} + \frac{\eta^\gamma}{t^{\gamma-1}} = 0$. The work of Emden also got the attention of physicists outside the field of astrophysics. For instance, the works of Thomas [164] and Fermi

[40] resulted in the *Thomas-Fermi* equation, used in atomic theory (see below for more details). The terminology is somehow confusing. Occasionally (4.2) is also called the *Lienard-Emden* equation and equation (4.4) mentioned below is called the *Emden-Fowler* equation (no matter what the sign is). Equation (4.4) with "+" is sometimes called the Fowler equation. All these equations are still extensively being investigated by physicists and mathematicians, and lot of other applications is known. Some people in the current literature (which is related to our topic) call the equation

$$(r(t)\Phi_\alpha(y'))' + \delta p(t)\Phi_\gamma(y) = 0, \tag{4.3}$$

where $p(t) > 0$ and $\delta \in \{\pm 1\}$, as of the *Emden-Fowler type* when $\delta = 1$ and as of the *Thomas-Fermi type* when $\delta = -1$. The same terminology is used for corresponding higher order equations or systems.

As already indicated, equation (4.1) can be understood also as a natural generalization of half-linear equation (3.1).

If, in (4.3), $\alpha = 1$ and $0 < \gamma < 1$, then we speak about *sub-linear* equation (at infinity), while if $\alpha = 1$ and $\gamma > 1$, then we speak about *super-linear* equation. For general $\alpha, \gamma \in (0, \infty)$ we call (4.3) *sub-half-linear* or *sub-homogeneous* provided $\alpha > \gamma$, while we call it *super-half-linear* or *super-homogeneous* when $\alpha < \gamma$. Analogously we use this terminology for related higher order equations and systems, and also for the objects where nonlinearities are somehow close to power functions.

As one of the motivations for considerations in this chapter can be the Thomas-Fermi atomic model described by the following nonlinear singular boundary value problem

$$y'' = \frac{1}{\sqrt{t}} y^{3/2},$$
$$y(0) = 1, \ y(\infty) = 0,$$

see Thomas [164], Fermi [40]; it is in fact a dimensionless form of the radially symmetric Poisson equation. As already mentioned, more general equation of physical interest is the Emden-Fowler one

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(s^\varrho \frac{\mathrm{d}u}{\mathrm{d}s}\right) \pm s^\sigma u^\gamma = 0, \tag{4.4}$$

where $\varrho, \sigma, \gamma \in \mathbb{R}$. For $\varrho \neq 1$, it is reducible to the form $y'' \pm t^\tau y^\gamma = 0$, where τ depends on ϱ, σ, γ, and for $\varrho = 1$, to the form $y'' \pm e^{(\sigma+1)t} y^\gamma = 0$, see Bellman [12, Chapter 7]. It is perhaps worthwhile mentioning Fowler's statement that even for $p(t) \sim t^\sigma$ as $t \to \infty$ (instead of $p(t) = t^\sigma$) his method is not applicable.

Before we come to connections with regular variation, note for instance that Kamo and Usami in [70, 71] consider the equation

$$(\Phi_\alpha(y'))' = p(t)\Phi_\gamma(y), \tag{4.5}$$

where $\alpha, \gamma > 0$ and $p(t) \sim t^\sigma$ as $t \to \infty$. Under (natural) additional conditions on α, γ, σ, they show that solutions y of (4.5) in a certain basic asymptotic class have the form $y(t) \sim Kt^{\delta}$, where $K = K(\alpha, \gamma, \sigma)$, $\delta = \delta(\alpha, \gamma, \sigma)$. The key role is played by the asymptotic equivalence theorem which says, roughly speaking: If the coefficients of two equations of the form (4.5) are asymptotically equivalent, then their solutions which are "of the same type" are also asymptotically equivalent. Of course, it is the equation with t^{σ} as the coefficient which is used for comparison purposes.

Realizing now that $\mathcal{RV}$ functions can be understood as a (nontrivial) extension of functions asymptotically equivalent to power ones, we can ask natural questions: How about an extension in the sense that the coefficient in the Emden-Fowler equation is a regularly varying function? Or, how about an extension in the sense of replacing the nonlinearity in the Emden-Fowler equation by a regularly varying function? All these and also further related problems are discussed in this chapter. Note for example, that such results as the above asymptotic equivalence theorem cannot be used in a general case. We will describe various approaches, which are used either in the same settings or in different settings.

4.2 Former results

Before we present a survey of recent results, let us recall some classical works made in particular by Avakumović, Geluk, Marić, and Tomić.

4.2.1 Superlinear second order equations in the work by Avakumović and related results

A substantial generalization of considerations about equation (4.4) was made by Avakumović in 1947, [9]. In fact, it is the first paper connecting regular variation and differential equations. It deals with the special type of (4.1), namely

$$y'' = p(t)y^{\gamma}, \tag{4.6}$$

where $p \in \mathcal{RV}(\sigma)$ is a continuous function and $\gamma > 1$. He proved the following statement.

Theorem 4.1 (Avakumović [9])**.** *Let $p \in \mathcal{RV}(\sigma)$ with $\sigma > -2$ and $\gamma > 1$. If y is an eventually positive solution of* (4.6) *such that* $\lim_{t\to\infty} y(t) = 0$, *then*

$$y(t) \sim \left(\frac{(1+\gamma+\sigma)(\sigma+2)}{(\gamma-1)^2}\right)^{\frac{1}{\gamma-1}} \left(t^2 p(t)\right)^{-\frac{1}{\lambda-1}}.$$

The proof is rather involved and proceeds by considering a suitable function $h(t)$, satisfying the relations $t^{\sigma+\gamma}L_p(t)h^{\gamma-1}(t) \sim (1+\gamma+\sigma)(2+\sigma)$ (with $L_p(t) = p(t)/t^{\sigma}$) and $h''(t) \sim p(t)h^{\gamma}(t)$, and applying a "variation of constants" $y(t) = h(t)z(\varphi(t))$, where φ is a solution of $h\varphi'' + 2h'\varphi' + \delta h\varphi'^2 = 0$, $\delta > 0$. The resulting differential equation for z is of the form $z_{\varphi\varphi} - \delta z_{\varphi} = f(t)z(g(t)z^{\gamma-1} - 1)$, where $f(t) \geq \zeta > 0$ and $g(t) \sim 1$. Tauberian considerations show that $z(\varphi(t)) \sim 1$.

A refinement of the last asymptotic formula is made in Avakumović [10] along with applications which lead to refined asymptotic formulae for $o(1)$ solutions of (4.6).

Only in 1991, Geluk in [46] presented a simple and elegant proof using a result on smoothly varying functions (see (1.11)) proved by Balkema, de Haan, and Geluk.

Geluk proved, in fact, the following statement.

Theorem 4.2. *Let $p \in \mathcal{RV}(\sigma)$ with $\sigma > -2$ and $\gamma > 1$. If y is an eventually positive and bounded solution of* (4.6)*, then $y \in \mathcal{RV}(-(\sigma+2)/(\gamma-1))$.*

Proof. Substitutions $u = y^{1-\gamma}$ and $v(t) = \ln u(e^t)$ show that v satisfies the equation

$$v'' - v' - \beta v'^2 = -e^{\psi - v},$$

where $\psi = \ln[(\gamma-1)e^{2t}p(e^t)]$, $\beta = 1/(\gamma-1)$. By the observations at (1.11), applied to the function $(\gamma-1)t^2p(t)$, there exists a function $\psi_1(t)$ such that $\psi(t) - \psi_1(t) \to 0$, $\psi_1'(t) \to \sigma+2$, $\psi_1''(t) \to 0$ as $t \to \infty$, and $\psi_1(t) \le \psi(t)$ for t sufficiently large. By substituting $v(t) = \psi_1(t) + h(t)$, the previous equation is reduced to

$$h'' - \omega h' - \beta h'^2 = -(1+o(1))e^{-h} + (\sigma+2)(1+\beta\sigma+2\beta) + o(1)$$

with $\omega(t) \to 2\beta(\sigma+2)+1$ as $t \to \infty$. We claim that $h(t)$ tends to a finite limit as $t \to \infty$. The following three cases are possible.

(i) $h'(t) > 0$ for $t > t_0$. Then h is ultimately increasing and its limit exists. If $h(t) \to \infty$, then by the preceding asymptotic equation, for $t \ge t_0$, one has $h'' > \omega h' + \beta h'^2 > h'/2$. This implies $h'(t) \to \infty$, and so, due to the mentioned equation, $(-1/h'(t))' \to \beta$ as $t \to \infty$. Hence, integrating, $-1/h'(t) \sim \beta t$, which contradicts the assumption $h'(t) > 0$ for large t.

(ii) $h'(t) < 0$ for $t > t_0$. Then h is ultimately decreasing and its limit exists. The case when $h(t) \to -\infty$ as $t \to \infty$ is again disposed of. Because of $\psi_1 \le \psi$, the equation for v gives

$$-v'' + v' + \beta v'^2 = e^{\psi - v} \ge e^{\psi_1 - v} = e^{-h}.$$

Since $-h(t) \to \infty$ as $t \to \infty$, there exists a sequence $\{t_n\}$ such that $v'(t_n) \to \pm\infty$ as $n \to \infty$. If $v'(t_n) \to \infty$, then $h'(t_n) \to \infty$, a contradiction. The case $v'(t_n) \to -\infty$ implies $u'(\exp t_n) < 0$, hence $y'(\exp t_n) > 0$ for n sufficiently large. Since $y''(t) > 0$, this contradicts the boundedness of y.

(iii) $h'(t)$ oscillates. This implies the existence of a sequence $\{t_n\}$ such that $h'(t_n) = 0$ and $t_n \to \infty$ as $n \to \infty$. If $h''(t_n) < 0$ (i.e., $h(t)$ attains its maximum for t_n), then for large n one has $h(t_n) < -\ln[(\sigma+2)(1+\beta\sigma+2\beta)]$. Similarly if $h''(t_n) > 0$, we find $h(t_n) > -\ln[(\sigma+2)(1+\beta\sigma+2\beta)]$, a contradiction.

Thus $h(t)$ tends to a finite constant as $t \to \infty$ which then implies $t^2p(t) \sim ky^{1-\gamma}$ as $t \to \infty$. Hence, y is regularly varying of index $-(\sigma+2)/(\gamma-1)$. □

Remark 4.1. The conclusion $y \in \mathcal{RV}(-(\sigma+2)/(\gamma-1))$ implies that $y(t) \to 0$. Moreover, y'' is regularly varying as the product of two $\mathcal{RV}$ functions. Application of Karamata's theorem then gives the exact value of the constant k, and thus we obtain the asymptotic formula from Theorem 4.1.

4.2.2 Superlinear second order equations in the former works by Marić and Tomić

In this subsection we present results based on Marić [105] and Marić, Tomić [110, 111, 112].

We continue in considerations from the previous subsection. Neither Avakumović nor Geluk consider the border case $\sigma = -2$ when the solutions tending to zero may still exists (see e.g. Wong [168], Taliaferro [158]). Consider the equation

$$y'' = p(t)F(y), \tag{4.7}$$

where $p(t)$ and $F(y)$ are continuous and positive for $t > 0$ and $y > 0$. Wong's result reads as follows. Let $F(t)/t$ increase. Then (4.7) has a positive solution which tends to zero if and only if

$$\int^{\infty} sp(s)\,\mathrm{d}s = \infty.$$

The above monotonicity condition is changed to

$$\limsup_{u\to 0}\sup_{0<\lambda<d}\frac{F(\lambda u)}{\lambda F(u)} < 1$$

for some $d > 0$, by Taliaferro, not affecting the statement of the theorem.

Marić and Tomić (see [105, 110, 111, 112]) in their consideration neither the monotonicity nor the Taliaferro condition need always to hold, in which case the condition $\int^{\infty} sp(s)\,\mathrm{d}s = \infty$ is assumed to hold independently (and the existence of relevant solution as well). First we present the result concerning the estimates of solutions.

Theorem 4.3. *Let $p \in \mathcal{RB}$ and such that*

$$\int^{\infty} sp(s)\,\mathrm{d}s = \infty. \tag{4.8}$$

Assume that $F \in \mathcal{RB}_0$ and such that

$$u^{-\gamma}F(u) \quad \textit{almost decreases for some } \gamma > 1 \tag{4.9}$$

as $u \to 0$. Then for every eventually positive solution y of (4.7) *tending to zero there holds*

$$\frac{F(y(t))}{y(t)} \asymp \frac{1}{\int_a^t sp(s)\mathrm{d}s} \tag{4.10}$$

as $t \to \infty$. Moreover, $F(u(t))/u(t) \in \mathcal{RB}$.

Proof. Since the functions p and F are regularly bounded, in addition to hypotheses (4.8) and (4.9), due to a relation of $\mathcal{RB}$ with almost monotonicity, there hold

$$t^{\delta}p(t) \quad \text{almost increases for some } \delta, \tag{4.11}$$

$$t^{\omega}p(t) \quad \text{almost decreases for some } \omega < \delta \tag{4.12}$$

for large t. Inequalities which yield (4.10) are then the results of a series of various estimates. This part of the proof is rather technical; for details see [105]. □

Notice that in view of the properties of $\mathcal{RV}$ functions, if $p \in \mathcal{RV}(-2)$, then the integral $I(t) = \int_a^t sp(s)\,\mathrm{d}s$ is a new $\mathcal{SV}$ function which cannot be disposed of in general by estimating it in a unique way. If e.g. $p(t) = t^{-2}\ln t$, then $I(t) \sim \frac{1}{2}\ln^2 t$ as $t \to \infty$, whereas if $f(t) = 1/(t^2 \ln t)$, then $I(t) \sim \ln\ln t$ as $t \to \infty$. However this is possible by restricting the rate of decay of function p. More precisely, the following statement holds.

Corollary 4.1. *Let $p \in \mathcal{RB}$ and such that for large t*

$$t^{\delta}p(t) \text{ almost increases for some } \delta < 2.$$

Assume that $F \in \mathcal{RB}_0$ and such that

$$u^{-\gamma}F(u) \quad \text{almost decreases for some } \gamma > 1$$

as $u \to 0$. Then for every eventually positive solution y of (4.7) *tending to zero there holds*

$$\frac{F(y(t))}{y(t)} \asymp \frac{1}{t^2 p(t)}$$

as $t \to \infty$.

Proof. Since $\delta < 2$, condition (4.8) is fulfilled and the previous theorem applies. Furthermore, due to (4.11),

$$\int_a^t sp(s)\,\mathrm{d}s \le Mt^{\delta}p(t)\int_a^t s^{1-\delta}\mathrm{d}s \le t^2 p(t),$$

which gives the left-hand side of the required inequality. The right-hand one is obtained likewise by using (4.11) instead of (4.11). □

Next we describe the asymptotic behavior of solutions to (4.7), where we assume that instead of being regularly bounded, the functions p and F are in $\mathcal{RV}$ and $\mathcal{RV}_0$, respectively. Then instead of estimates for large t we obtain a precise asymptotic behavior of the function $F(y(t))/y(t)$ as $t \to \infty$. We also tacitly assume that Wong's condition (4.8) holds in order not to violate the existence of the considered solutions y tending to zero even in the simplest cases e.g. such as $y'' = t^2 L_p(t) y^{\gamma}$.

Theorem 4.4. *Assume*

$$p(t) = t^{\sigma} L_p(t), \ \ \sigma \geq -2, \ \ F(u) = u^{\gamma} L_F(u), \ \ \gamma > 1,$$

where $L_p \in \mathcal{SV}$ *and* $L_F \in \mathcal{SV}_0$. *Then for every solution* $y(t)$ *of* (4.7) *tending to zero as* $t \to \infty$ *there holds for* $t \to \infty$:

(i) for $\sigma > -2$

$$y^{\gamma-1}(t) L_F(y(t)) \sim \frac{(1+\sigma+\gamma)(2+\sigma)}{(\gamma-1)^2} \cdot \frac{1}{t^{2+\sigma} L_p(t)}$$

and $y \in \mathcal{RV}((\sigma+2)/(1-\gamma))$;

(ii) for $\sigma = -2$

$$y^{\gamma-1}(t) L_F(y(t)) \sim \frac{1}{(\gamma-1)\int_a^t L_p(s)/s \, \mathrm{d}s}$$

and $y \in \mathcal{SV}$.

Proof. The proof is rather technically complicated. We briefly mention only the main ideas. The details can be found in [105, 112]. Put

$$\begin{aligned} I(t) &:= \int_a^t \frac{1}{s^2} \int_a^s u^2 p(u) \, \mathrm{d}u \, \mathrm{d}s, \\ G(y) &:= \int_0^y \frac{1}{t} \int_0^t \frac{F(u)}{u^2} \, \mathrm{d}u \, \mathrm{d}t, \\ Z(t) &:= I(t) G(y(t)). \end{aligned}$$

The function Z satisfies the differential equation

$$\frac{Z''}{Z} \cdot \frac{I}{I'} = 2\frac{Z'}{Z} + \frac{I''}{I'} - \frac{2I'}{I} + Z\frac{p}{I'} \cdot \frac{FG'}{G^2} + \left(\frac{Z'}{Z} - \frac{I'}{I}\right)^2 \frac{I}{I'} \cdot \frac{GG''}{G'^2}.$$

The behavior of all intervening functions (coefficients) is then determined. Among others, it is shown that

$$G(y) \sim \frac{F(y)}{(\gamma-1)^2 y}$$

as $y \to 0$,

$$I(t) \sim t^2 p(t)/((\sigma+3)(\sigma+2))$$

as $t \to \infty$, and

$$\lim_{t\to\infty} Z(t) = c > 0.$$

In addition to various estimates and properties of $\mathcal{RV}$ functions, also the Avakumović theorem (Theorem 4.1) and estimates (4.10) find application in proving the above estimations. The asymptotic formula in the part (i) is then the result of these relations and some additional observations; among others we apply the following claim: Since $H(y) = y^{\gamma-1} L_F(y)$ is in $\mathcal{RV}_0(\gamma-1)$, we find $\tilde{H} \in \mathcal{RV}_0(1/(\gamma-1))$ such that $\tilde{H}(H(y)) \sim y$. Concerning the part (ii), in contrast to the part (i), first it is proved that $y \in \mathcal{SV}$ and then the asymptotic formula is derived (using analogous arguments to those in the part (i)). □

Remark 4.2. The natural question of how to obtain asymptotic behavior of solutions y by inverting formulae in Theorem 4.4 turns out to be in general a very difficult one (but not related to differential equations). In fact, that theorem gives only that solutions are of the form

$$y(t) = \begin{cases} t^{(\sigma+2)/(1-\gamma)} L_1(t) & \text{for } \sigma > -2 \\ L_2(t) & \text{for } \sigma = -2 \end{cases}$$

where $\mathcal{SV}$ functions L_1, L_2 satisfy the following implicit asymptotic relations

$$L_1^{\gamma-1}(t) L_F\Big(t^{(\sigma+2)/(1-\gamma)} L_1(t)\Big) \sim \frac{(1+\sigma+\gamma)(2+\sigma)}{(\gamma-1)^2} \cdot \frac{1}{L_p(t)},$$

$$L_2^{\gamma-1}(t) L_F(L_2(t)) \sim \frac{1}{(\gamma-1)\int_a^t L_p(s)/s \,\mathrm{d}s}$$

as $t \to \infty$.

4.2.3 Rapidly varying coefficient and nonlinearity

In the previous subsection it is assumed for the functions p and F in (4.7) to be $\mathcal{RV}$ and $\mathcal{RV}_0$, respectively. Therefore that analysis did not include even such simple cases as $p(t) = e^t$ or $F(u) = e^{-1/u}$. These examples indicate that we intend to keep as the main feature of equation (4.7) — to be superlinear and such that $F(u) \to 0$ as $u \to 0$. This type of problem was considered in Marić [105], see also Marić, Radaśin [106, 107, 108].

Let us write equation (4.7) as

$$y'' = f(g(t))\varphi(\psi(t)). \tag{4.13}$$

We assume that $g(t)$ is positive, increasing to infinity as $t \to \infty$, twice differentiable, and such that there exists the limit

$$\lim_{t\to\infty} \frac{g(t)g''(t)}{g'^2(t)} = A.$$

For $\psi(u)$ we assume that it is positive, decreasing to zero as $u \to 0$, twice differentiable, and such that there exists the limit

$$\lim_{u\to 0} \frac{\psi(u)\psi''(u)}{\psi'^2(u)} = B.$$

Recall that then $A, B \le 1$; if $A < 1$, then $g \in \mathcal{RV}(1/(1-A))$; if $A = 1$, then $g \in \mathcal{RPV}$; if $B < 1$, then $\psi \in \mathcal{RV}_0(1/(1-B))$; if $B = 1$, then $\psi \in \mathcal{RPV}_0$.

Theorem 4.5. *Let $f \in \mathcal{RV}(\sigma)$ with $\sigma + 2 - 2A \geq 0$ for $A \neq 1$ and $\sigma > 0$ for $A = 1$, and $\varphi \in \mathcal{RV}_0(\gamma)$ with $\gamma > 1 - B$.. Assume that A and B are not both equal to one. Then for all positive solutions $y(t)$ of* (4.13) *which tend to zero as $t \to \infty$, there holds*

$$\frac{\psi'(y(t))}{\psi(y(t))}\varphi[\psi(y(t))] \sim \frac{(1-A)(\gamma+B-1)+(1-B)(\sigma+2-2A)}{(\gamma+B-1)^2\int_a^t \frac{g(s)}{g'(s)} f(g(s))\,\mathrm{d}s}$$

as $t \to \infty$.

Proof. Basically, the proof runs along the same line as the proof of Theorem 4.4, except that here we do not have estimates as (4.10), which requires some additional arguments. We put

$$\begin{aligned} I(t) &:= \int_a^t \left(\frac{g'(s)}{g(s)}\right)^2 \int_a^s \left(\frac{g(u)}{g'(u)}\right)^2 f(g(u))\,\mathrm{d}u\,\mathrm{d}s, \\ G(y) &:= \int_0^y \frac{\psi'(s)}{\psi(s)} \int_0^s \left(\frac{\psi(u)}{\psi'(u)}\right)^2 \varphi(\psi(u))\,\mathrm{d}u\,\mathrm{d}s, \\ Z(t) &:= I(t)G(y(t)). \end{aligned}$$

The function Z satisfies the differential equation

$$\frac{Z''}{Z}\cdot\frac{I}{I'} = 2\frac{Z'}{Z} + \frac{I''}{I'} - \frac{2I'}{I}\cdot\frac{FG'}{G^2} + \left(\frac{Z'}{Z} - \frac{I'}{I}\right)^2 \frac{I}{I'}\cdot\frac{GG''}{G'^2}.$$

For details see [105]. □

Remark 4.3. Somehow related results were obtained by Taliaferro (see [158, 160] and also [105, Section 3.6]) for the more general equation $y'' = F(t, y, y')$. He uses some definitions which are — as stated there — "partially motivated" by $\mathcal{RV}$ functions, but his methods make no use of Karamata functions which is the subject of this treatise.

4.3 Selection of recent results

4.3.1 Introductory remarks

In the last decade, many papers have appeared (and still are appearing) which are devoted to the investigation of various forms of Emden-Fowler type equations (incl. systems and higher order equations) in the framework of regular variation. It is impossible to present here all these results. Instead of this we prefer to make a reasonable selection of typical results which will show a wide variety of methods which are employed — this is in accordance with the principal aim of our treatise.

It is worthy of note that there are some overlaps in the papers in spite of different methods. However, this fact is quite natural, when one realizes that a typical result is of the form: It the coefficients in an equation are $\mathcal{RV}$, then (at least on or all) solutions in certain basic asymptotic classes are $\mathcal{RV}$ (with the index which depends on the indices of coefficients and powers of nonlinearities) and their behavior is governed by a specific asymptotic formula.

Before we present selected results, let us try to give at least a brief (but comprehensive) survey of the literature devoted to this topic, that is investigation of quasilinear equations in the framework of regular variation.

First note that when we study $\mathcal{RV}$ (or somehow similar) solutions we consider nonoscillatory solutions. There is an extensive literature devoted to classifications of nonoscillatory solutions and investigation of the existence and behavior of solutions in the individual classes. One of the most important publications along this line is the book [79] by Kiguradze and Chanturia. Instead of quoting other works, note that some related references are spread in this text, in the places where we utilize the relevant results.

There are various approaches to examine Emden-Fowler type equations in the framework of regular variation and several independent groups which work on it.

Existence and asymptotic behavior of $\mathcal{RV}(1)$ increasing solutions of the second order Thomas-Fermi type equation

$$y'' = p(t)F(x),$$

$p(t) > 0$, in the sub-linear case were studied by Kusano, Manojlović, Marić in [83]; other $\mathcal{RV}$ solutions of the same problem are examined in [104] by Manojlović and Marić. The generalized Thomas-Fermi equation

$$(\Phi_\alpha(y'))' = p(t)\Phi_\gamma(y),$$

$p(t) > 0$, is considered in the sub-homogeneous case ($\alpha > \gamma$) by Kusano, Marić, Tanigawa in [100]; existence conditions and formulas for $\mathcal{SV}, \mathcal{RV}(1)$ solutions and for nearly $\mathcal{RV}$ solutions are established. An asymptotic analysis of $\mathcal{RV}$ solutions to the Emden-Fowler type equation

$$y'' + p(t)\Phi_\gamma(y) = 0,$$

$p(t) > 0$, is made in the sub-linear case in [84] and existence of $\mathcal{SV}, \mathcal{RV}(1)$ solutions is discussed for the same equation in both the sub- and super-linear cases in [85] by Kusano and Manojlović. The same authors study the sublinear Emden-Fowler type equation

$$y'' + p(t)F(x) = 0,$$

$p(t) > 0$, in [86]; existence and asymptotic behavior of $\mathcal{SV}, \mathcal{RV}(1)$ solutions are examined, and necessary conditions for the existence along with asymptotic formulas for intermediate solutions are established. A generalized Emden-Fowler type equation of the form

$$(r(t)\Phi_\alpha(y'))' + p(t)\Phi_\gamma(y) = 0,$$

$r(t), p(t) > 0$, $\alpha > \gamma$, is considered in [91] by Kusano, Manojlović, and Milošević and in [64] by Jaroš, Kusano and Manojlović; its intermediate generalized $\mathcal{RV}$ solutions are discussed. The fourth order sublinear equation

$$y^{(4)} + \delta p(t)\Phi_\gamma(y) = 0,$$

$p(t) > 0$, $0 < \gamma < 1$, is considered by Kusano and Manojlović in [87] under the condition $\delta = 1$ and in [88] under the condition $\delta = -1$; existence of (all possible) $\mathcal{RV}$ solutions and their accurate behavior are examined. This equation with $\delta = 1$ is examined also by Kusano, Manojlović, and Tanigawa in [63]. The more general equation

$$(\Phi_\alpha(y''))'' + p(t)\Phi_\gamma(y) = 0,$$

$p(t) > 0$, $\alpha > \gamma > 0$, is considered by the same authors in [93]; all possible types of positive solutions are examined. Jaroš, Kusano, and Tanigawa in [67] discuss the existence and asymptotic behavior of $\mathcal{RV}$ solutions of the third order sublinear equation

$$y''' + p(t)\Phi_\gamma(y) = 0,$$

$p(t) > 0$, see also the papers [68, 69] by the same authors. The two-dimensional system

$$x' = \delta p(t)y^\alpha, \quad y' = \delta q(t)x^\beta,$$

$p(t), q(t) > 0$, $\alpha, \beta > 0$, $\alpha\beta < 1$, $\delta = \pm 1$, is considered by Jaroš and Kusano in [61]; strongly monotone $\mathcal{RV}$ solutions are analyzed, see also [63]. The same authors study strongly monotone $\mathcal{RV}$ solutions of the second order system

$$x'' = p(t)y^\alpha, \quad y'' = q(t)x^\beta,$$

$p(t), q(t) > 0$, $\alpha, \beta > 0$, $\alpha\beta < 1$, see [62], and strongly decreasing solutions of the system

$$x'' = p_1(t)x^{\alpha_1} + q_1(t)y^{\beta_1}, \quad y'' = p_2(t)x^{\alpha_2} + q_2(t)y^{\beta_2},$$

where $p_1, q_1, p_2, q_2 \in \mathcal{RV}$ and $\alpha_1, \beta_1, \alpha_2, \beta_2 > 0$, as presented by Tanigawa at the conference Equadiff 13. Kusano and Manojlović consider the odd-order equation

$$y^{(2n+1)} + \delta p(t)\Phi(\gamma) = 0,$$

$p(t) > 0$, $0 < \gamma < 1$. Existence and asymptotic behavior of all possible types of positive solutions to this equation is studied in the framework of regular variation; the case $\delta = 1$ in [89] and the case $\delta = -1$ in [90].

In many results of the above mentioned papers, various modifications of the following technique is utilized: Certain asymptotic relation is investigated which can be considered as an "approximation" of the given differential equation rewritten to a certain integral form; properties of $\mathcal{RV}$ functions — mainly the Karamata integration theorem — are then extensively used there.A priori bounds are obtained. Further, the Schauder-Tychonoff fixed point theorem in locally convex spaces plays an important role. For some of the equations there has been made a complete analysis of the existence and asymptotic behavior of regularly varying solutions which belong to all possible Kiguradze type classes. Various utilizations of this technique are presented in Subsections 4.3.2, 4.3.3, 4.3.4, and 4.3.6.

A somehow different approach is represented by the works of Evtukhov and others. Note that — in some of these works — the conditions which are usually imposed on the coefficients and nonlinearities in the equation are not directly in terms of regular variation (or rapid variation), but in fact they belong to these classes (or are close to them) due to known results (like the Karamata theorem) which enables an alternative expression. Similar observations hold for some of the classes of solutions which are studied. It is worthy of note that some of these papers (especially the older ones) do not use (or even mention) the concept of regular variation at all, even though the considerations are "close" to regular variation. Typically, both types of equations (Emden-Fowler and Thomas Fermi) are simultaneously studied and also both the cases (sub- and super-linearity) are considered. Even a negative power or index in a nonlinearity is sometimes allowed. The conditions for the existence of solutions in various asymptotic classes (they are called P_ω and have various modifications) are derived and asymptotic representations are established. Evtukhov and Samoilenko in [36] consider the equation

$$y^{(n)} = \delta p(t) F(t),$$

$p(t) > 0, \delta = \pm 1$. The second order case is studied by Evtukhov and Kharkov in [34] and by Evtukhov and Abu Elshour in [32], and the third order case by Evtukhov and Stekhun [37]. Bilozerowa and Evtukhov in [13] examine the generalization of the Emden-Fowler equation in the following form

$$y^{(n)} = \delta p(t) \prod_{i=0}^{n-1} F_i(y^{(i)}),$$

$p(t) > 0, \delta = \pm 1$; the second order case is considered in [33]. The equation

$$y'' = \sum_{i=1}^{m} \delta_i p_i(t)(1 + q_i(t)) F_{i0}(y) F_{i1}(y')$$

is studied by Kozma in [82]. Evtukhov and Vladova in [39] concentrate on the cyclic system

$$y_i' = \delta_i p_i(t) F_i(y_{i+1}),$$

$i = 1, \dots, n$, $p_i(t) > 0$, $\delta_i = \pm 1$, y_{n+1} means y_1; the two dimensional case is studied in [38]. Subsection 4.3.7 offers a more detailed description of the method used by Evtukhov et al.

Another direction of the approach in the study of objects related to Emden-Fowler type equations and regular variation is represented by the following three works. A substantial part of these works is devoted to a generalization of the results described after (4.5). In other words, typical feature is that not only the existence of $\mathcal{RV}$ solutions is studied and asymptotic formulas are established, but, in addition, $\mathcal{RV}$ behavior of all solutions in a given basic asymptotic class of solutions is guaranteed. Only some of the Kiguradze type subclasses have been examined in such a way, namely the most extreme ones. Matucci and Řehák in [123] study decreasing $\mathcal{RV}$ solutions of the coupled system

$$\begin{cases}(p(t)\Phi_\alpha(x'))' = \varphi(t)\Phi_\lambda(y),\\ (q(t)\Phi_\beta(y'))' = \psi(t)\Phi_\mu(x),\end{cases}$$

$p(t), r(t), \varphi(t), \psi(t) > 0$, $\alpha\beta > \lambda\mu$, see Subsection 4.3.5 for couple of notes. Strongly monotone solutions of the system

$$y_i' = \delta p_i(t) F_i(y_{i+1}),$$

$i = 1, \dots, n$, $p_i(t) > 0$, $\delta = \pm 1$, y_{n+1} means y_1, are investigated by Řehák in [146] and by Matucci and Řehák in [124]. See Subsection 4.3.8 where this approach is discussed in details. A slight modification of the setting of the latter paper enables us to include also equations with a general Φ-Laplacian, see Subsection 5.1.2.

4.3.2 Asymptotic behavior of $\mathcal{SV}$ and $\mathcal{RV}(1)$ solutions of sublinear second order equations

The result in this section is selected from the paper [86] by Kusano and Manojlović as a representant of the below described method. Consider the second order Emden-Fowler type equation

$$y'' + p(t)F(y) = 0, \tag{4.14}$$

where $p : [a, \infty) \to (0, \infty)$ is a continuous function with $p \in \mathcal{RV}(\sigma)$, $\sigma \in \mathbb{R}$, and $F : (0, \infty) \to (0, \infty)$ is an increasing continuous function with $F \in \mathcal{RV}(\gamma)$. The sublinearity condition $0 < \gamma < 1$ is assumed. Recall that the following generalization of the known Belohorec theorem holds: Equation (4.14) has a positive solution if and only if $\int_a^\infty p(s)F(s)\,\mathrm{d}s < \infty$.

Suppose that (4.14) has a positive solution y (called *intermediate*) such that

$$\lim_{t\to\infty} \frac{y(t)}{t} = 0 \quad \text{and} \quad \lim_{t\to\infty} y(t) = \infty. \tag{4.15}$$

From the conventional classification of eventually positive solutions, in addition to intermediate solutions, only the two following (and somehow easier) classes are possible: $\lim_{t\to\infty} y(t) = \text{const} > 0$ (the so-called *minimal* solutions) or $\lim_{t\to\infty} x(t)/t = \text{const} > 0$ (the so-called *maximal* solutions); the terminology may come from the fact that a positive solution y always satisfies $c_1 \le x(t) \le c_2 t$, $t \ge T_x$, for some positive constants $c_1, c_2 \in \mathbb{R}$. It is clear that a minimal solution is in tr-$\mathcal{SV}$ and a maximal solution is in tr-$\mathcal{RV}(1)$. However, as we will see, there can also be nontrivial $\mathcal{SV}$ or nontrivial $\mathcal{RV}(1)$ solutions which, of course, must be sought among intermediate solutions. Note that in the sub-linear case, the sufficient and necessary condition for the existence of a solution y satisfying (4.15) reads as

$$\int_a^\infty p(s)F(s)\,\mathrm{d}s < \infty \quad\text{and}\quad \int_a^\infty sp(s)\,\mathrm{d}s = \infty,$$

see Kusano, Ogata, Usami [101]. Since $y'(\infty) = 0$, integrating (4.14) first from t to ∞ and then from a to t, we have

$$y(t) = y(a) + \int_a^t \int_s^\infty p(\tau)F(y(\tau))\,\mathrm{d}\tau\,\mathrm{d}s, \tag{4.16}$$

$t \geq a$. It is somehow natural to search for a solution of (4.14) with specific asymptotic behavior at infinity as a fixed point of the integral operator

$$\mathcal{F}y(t) = C + \int_a^t \int_s^\infty p(\tau)F(y(\tau))\,\mathrm{d}\tau\,\mathrm{d}s,$$

$t \geq a$, in some suitably chosen set $\mathcal{X}$ of $C[a, \infty)$. A thorough analysis can be made of the existence and the precise asymptotic behavior of $\mathcal{RV}$ solutions of the integral asymptotic relation

$$y(t) \sim \int_a^t \int_s^\infty p(\tau)F(y(\tau))\,\mathrm{d}\tau\,\mathrm{d}s, \quad t \to \infty, \tag{4.17}$$

which can be considered as an "approximation" of (4.16). Note that (4.17) follows from (4.16), using that $y(\infty) = \infty$. Then the set $\mathcal{X}$ with the required properties can be found by means of $\mathcal{RV}$ solutions of the integral asymptotic relation (4.17).

It is worthy of note that an important role in the proof is played by the fact that the auxiliary linear second order equation possesses $\mathcal{SV}$ and $\mathcal{RV}(1)$ solutions. More precisely, we will utilize the following statement which follows from Theorem 2.2 and a simple application of the properties of $\mathcal{RV}$ functions to the (positive) solution $y(t) = c_1y_1(t) + c_2y_2(t)$, $c_1, c_2 \in \mathbb{R}$, $\{y_1, y_2\}$ being a fundamental set of $\mathcal{RV}$ solutions.

Lemma 4.1. *If*

$$\lim_{t\to\infty} t \int_t^\infty q(s)\,\mathrm{d}s = 0,$$

then every (eventually) positive solution of the equation $y'' + q(t)y = 0$, $q(t) > 0$, is $\mathcal{SV}$ or $\mathcal{RV}(1)$.

We now give the conditions guaranteeing the existence of nontrivial $\mathcal{SV}$ and $\mathcal{RV}(1)$ solutions and establish asymptotic formulae.

Theorem 4.6. *Suppose that $p \in \mathcal{RV}(\sigma)$ and $F \in \mathcal{RV}(\gamma)$ with $\gamma \in (0,1)$.*

(i) Equation (4.14) *possesses nontrivial $\mathcal{SV}$ solutions if and only if $\sigma = -2$ and $\int_a^\infty sp(s)\,\mathrm{d}s = \infty$. The asymptotic behavior of each nontrivial $\mathcal{SV}$ solution y is governed by the unique formula*

$$y(t) \sim F^{-1}\left(\int_a^t sp(s)\,\mathrm{d}s\right)$$

as $t \to \infty$.

(ii) Let, moreover,

$$F(tu(t)) \sim F(t)u^{\gamma}(t) \tag{4.18}$$

as $t \to \infty$, for every $u \in \mathcal{SV} \cap C^1$. Equation (4.14) *possesses nontrivial $\mathcal{RV}(1)$ solutions if and only if $\sigma = -\gamma - 1$ and $\int_a^\infty p(s)F(s)\,\mathrm{d}s < \infty$. The asymptotic behavior of any nontrivial $\mathcal{RV}(1)$ solution y is governed by the unique formula*

$$y(t) \sim t\left((1-\gamma)\int_t^\infty p(s)F(s)\,\mathrm{d}s\right)^{\frac{1}{1-\gamma}}$$

as $t \to \infty$.

Proof. We give the proof only of the "if" part. The "only if" part is proved in [86] and extensively uses the Karamata integration theorem and some basic properties of $\mathcal{RV}$ functions. Let $x_0(t), x_1(t)$ be the functions on $[a,\infty)$ defined by

$$x_0(t) = F^{-1}\left(\int_a^t sp(s)\,\mathrm{d}s\right),$$

$$x_1(t)t\xi_1^{\frac{1}{1-\gamma}}(t), \quad \text{where} \quad \xi_1(t) = (1-\gamma)\int_t^\infty p(s)F(s)\,\mathrm{d}s \in \mathcal{SV}.$$

The proof will be performed simultaneously and in two steps. In the first step we show that (4.14) possesses intermediate solution $y(t)$ satisfying $kx_0(t) \le y(t) \le Kx_0(t)$ or $kx_1(t) \le y(t) \le Kx_1(t)$, $t \ge T$, for some positive constants $T > a, k < 1, K > 1$ depending on whether the assumptions in (i) or (ii) are satisfied, respectively. Then, in the second step, with the help of Lemma 4.1, we show that solutions constructed in the first step are nontrivial $\mathcal{SV}$, resp. $\mathcal{RV}(1)$. Since we perform the proofs for $x_i(t)$. $i = 0,1$, simultaneously, the subscripts $i = 0,1$ will be deleted in the rest of the proof.

Step 1. It is not difficult to show — the Karamata integration theorem plays an important role and we use (4.18) — that x satisfies the asymptotic relation

$$x(t) \sim \int_a^t \int_s^\infty p(\tau)F(x(\tau))\,\mathrm{d}\tau\,\mathrm{d}s \tag{4.19}$$

as $t \to \infty$. Let K, k be fixed positive constants such that $K^{1-\gamma} \ge 4$ and $k^{1-\gamma} \le 1/2$. Note that $k < 1$ and $K > 2$. Using that $F(Kx(t)) \sim K^{\gamma}F(x(t))$, from (4.19) we have

$$\int_a^t \int_s^\infty p(\tau)F(Kx(\tau))\,\mathrm{d}\tau\,\mathrm{d}s \sim K^{\gamma}x(t)$$

as $t \to \infty$, which implies the existence of $T_0 > a$ depending only on K such that

$$\int_{T_0}^{t} \int_{s}^{\infty} p(\tau) F(Kx(\tau))\, \mathrm{d}\tau\, \mathrm{d}s \le 2K^{\gamma} x(t),$$

$t \ge T_0$. Let such a T_0 be fixed. We may assume that $x(t)$ is increasing on $[T_0, \infty)$. Since $x(t) \to \infty$ and $F(kx(t)) \sim k^{\gamma} F(x(t))$ as $t \to \infty$, from (4.19) we have

$$\int_{T_0}^{t} \int_{s}^{\infty} p(\tau) F(kx(\tau))\, \mathrm{d}\tau\, \mathrm{d}s \sim k^{\gamma} x(t)$$

as $t \to \infty$, and so there exists $T_1 > T_0$ depending only on k such that

$$\int_{T_0}^{t} \int_{s}^{\infty} p(\tau) F(kx(\tau))\, \mathrm{d}\tau\, \mathrm{d}s \sim \frac{k^{\gamma}}{2} x(t),$$

$t \ge T_1$. Let such a T_1 be fixed. Let us define $\mathcal{X}$ to be the set of continuous functions $y(t)$ on $[T_0, \infty)$ satisfying

$$\begin{cases} x(T_0) \le y(t) \le Kx(t) & \text{for } T_0 \le t \le T_1, \\ kx(t) \le y(t) \le Kx(t) & \text{for } t \ge T_1. \end{cases} \tag{4.20}$$

It is clear that $\mathcal{X}$ is a closed convex subset of the locally convex space $C[T_0, \infty)$ equipped with the topology of uniform convergence on compact subintervals of $[T_0, \infty)$. We now define the integral operator

$$\mathcal{F} y(t) = x(T_0) + \int_{T_0}^{t} \int_{s}^{\infty} p(\tau) F(y(\tau))\, \mathrm{d}\tau\, \mathrm{d}s,$$

$t \ge T_0$, and let it act on the set $\mathcal{X}$ defined above. It is not difficult to verify that $\mathcal{F}$ is a self-map on $\mathcal{X}$ and sends $\mathcal{X}$ continuously to a relatively compact subset of $C[T_0, \infty)$. Therefore we are able to apply the Schauder-Tychonoff fixed point theorem to conclude that there exists $y \in \mathcal{X}$ such that $y(t) = \mathcal{F} y(t)$, $t \ge T_0$. It is clear from (4.20) that y satisfies

$$kx(t) \le y(t) \le Kx(t), \tag{4.21}$$

$t \ge T_0$, which completes the proof of the first step.

Step 2. Let y be a solution obtained in the first step. It can be regarded as a solution of the linear differential equation $y'' + q(t)y = 0$ with $q(t) = p(t)\frac{F(y(t))}{y(t)}$. We will show that

$$\lim_{t\to\infty} t \int_{t}^{\infty} p(s) \frac{F(y(s))}{y(s)}\, \mathrm{d}s = 0. \tag{4.22}$$

Since y satisfies (4.21) it suffices to show that

$$\lim_{t\to\infty} t \int_{t}^{\infty} p(s) \frac{F(x(s))}{x(s)}\, \mathrm{d}s = 0.$$

Suppose first that the assumption of part (i) are satisfied. The using that $x(t) = x_0(t) \in \mathcal{SV}$ is a solution of the differential equation $x_0' = tp(t)F(x_0)$ and applying the L'Hospital rule we obtain

$$\lim_{t\to\infty} t\int_t^\infty p(s)\frac{F(x(s))}{x(s)}\,\mathrm{d}s = \lim_{t\to\infty}\frac{tx_0'(t)}{x_0(t)},$$

since for our $\mathcal{SV}$ x_0 the last limit is equal to 0. If the assumptions of part (ii) are satisfied, then $y(t) = x_1(t) \in \mathcal{RV}(1)$. Using (4.18) with application of the L'Hospital rule, we get

$$\begin{aligned}
\lim_{t\to\infty} t\int_t^\infty p(s)\frac{F(x(s))}{x(s)}\,\mathrm{d}s &= \lim_{t\to\infty} t\int_t^\infty s^{-\gamma-1}L_p(s)\frac{F(s)\xi_1^{\frac{\gamma}{1-\gamma}}(s)}{s\xi_1^{\frac{1}{1-\gamma}}(s)}\,\mathrm{d}s\\
&= \lim_{t\to\infty} t\int_t^\infty \frac{s^{-2}L_p(s)L_F(s)}{\xi_1(s)}\,\mathrm{d}s\\
&= \lim_{t\to\infty}\frac{L_p(t)L_F(t)}{\xi_1(t)}.
\end{aligned}$$

Since

$$\int_a^\infty \frac{L_p(s)L_F(s)}{s}\,\mathrm{d}s = \int_a^\infty p(s)F(s)\,\mathrm{d}s < \infty,$$

by the Karamata theorem we get

$$\lim_{t\to\infty}\frac{L_p(t)L_F(t)}{\int_t^\infty L_p(s)L_F(s)/s\,\mathrm{d}s} = 0,$$

and so (4.22) follows in both the cases (i) and (ii). It follows from Lemma 4.1 that $y \in \mathcal{SV}\cup\mathcal{RV}(1)$. But if the assumptions in (i) hold, noting that $x(t) = x_0(t) \in \mathcal{SV}$, we conclude that y must be $\mathcal{SV}$, and if the assumptions in (ii) hold, since $x(t) = x_1(t) \in \mathcal{RV}(1)$, solution y must be $\mathcal{RV}(1)$. □

Condition (4.18) amounts to requiring that $\mathcal{SV}$ part L_F of F satisfies

$$L_F(tu(t)) \sim L_F(t) \tag{4.23}$$

as $t \to \infty$ for every $u \in \mathcal{SV}\cap C^1$. Compare this condition with (4.76), (4.95), and (4.96). Many (usual) $\mathcal{SV}$ functions satisfy (4.23). For instance, $L_F(t) = \prod_{k=1}^N(\ln_k t)^{\alpha_k}$, $\alpha_k \in \mathbb{R}$, or L_F such that $L_F(t) \to c \in (0,\infty)$ as $t \to \infty$, etc. Condition (4.23) is not satisfied e.g. by $L_F(t) = \exp\left(\prod_{k=1}^N(\ln_k t)^{\beta_k}\right)$, $\beta_k \in (0,1)$.

4.3.3 $\mathcal{SV}$ solutions of second order super-linear equations

In this section which is based on Kusano, Manojlović [85] we discuss the existence and behavior of $\mathcal{SV}$ solutions to the equation

$$y'' + p(t)\Phi_\gamma(y) = 0, \tag{4.24}$$

where $p : [a, \infty) \to (0, \infty)$ is a continuous function. The super-linearity condition $\gamma > 1$ is assumed.

By the famous Atkinson result, (4.24) is oscillatory (i.e., all of its nontrivial solutions are nonoscillatory) if and only if

$$\int_a^\infty sp(s)\,\mathrm{d}s = \infty.$$

Recall that under the assumption of sub-linearity (i.e., $0 < \gamma < 1$), similar type of result was established by Belohorec, and reads as follows: Equation (4.24) is oscillatory if and only if

$$\int_a^\infty s^\gamma p(s)\,\mathrm{d}s = \infty.$$

Eventually positive solutions can be classified into the three types in the same way as in the previous section.

The proof of the main result heavily depends on a similar result for the linear differential equation

$$y'' + p(t)y = 0. \tag{4.25}$$

More precisely, first we form an infinite family of linear equations of the form (4.25) and then select with the help of fixed point techniques one equation from the family whose solution would give birth to the desired $\mathcal{RV}$ solution of Emden-Fowler equation (4.24). The feasibility of such a procedure is assured by the extensive use of the following statement which can be proved by means of the contraction mapping theorem. We define the mapping

$$\mathcal{F}v(t) = t\int_t^\infty \left(\frac{v(s) + Q(s)}{s}\right)^2 \mathrm{d}s$$

and consider it on the set

$$\left\{v \in C_0[T, \infty) : 0 \le v(t) \le \varphi(t), t \ge T\right\},$$

where Q, φ are defined below in the theorem, and $C_0[T, \infty)$ denotes the set of all continuous functions on $[T, \infty)$ which tend to zero.

Lemma 4.2 ([85]). *Assume that there is a continuous function $\varphi(t) : [a, \infty) \to (0, \infty)$ which is decreasing to 0 as $t \to \infty$ and such that $Q(t) \le \varphi(t)$ for large t, where Q is defined by*

$$Q(t) = t\int_t^\infty p(s)\,\mathrm{d}s.$$

Then equation (4.25) *has a $\mathcal{SV}$ solution y in the form*

$$y(t) = \exp\left\{\int_T^t \frac{v(s) + Q(s)}{s}\,\mathrm{d}s\right\},$$

$t \ge T$, for some $T > a$, in which $v(t)$ is a unique solution of the integral equation

$$v(t) = t \int_t^\infty \left(\frac{v(s) + Q(s)}{s} \right)^2 \mathrm{d}s,$$

$t \ge T$, and $0 \le v(t) \le 4\varphi^2(t)$ for only $t \ge T$.

We will use the notation

$$Q_L(t) = t \int_t^\infty p(s) L^{\gamma-1}(s)\,\mathrm{d}s.$$

Theorem 4.7. *Let $\gamma > 1$. Assume that $L \in \mathcal{SV}$ is such that*

$$L(t) = L(a) \exp\left\{ \int_a^t \frac{\delta_L(s)}{s} \mathrm{d}s \right\} \nearrow \infty \ \ \text{with } \delta_L(t) \searrow 0 \tag{4.26}$$

as $t \to \infty$. Suppose that there exists a constant $K > 0$ such that

$$Q_L(t) \le K\delta_L(t) \ \ \text{for large } t. \tag{4.27}$$

Then equation (4.24) possesses a $\mathcal{SV}$ solution y such that $y(t) \le L(t)$ for all large t.

Proof. First observe that due to (4.26) we have

$$\delta_L(t) = t\frac{L'(t)}{L(t)} > 0, \quad \int_a^\infty \frac{\delta_L(s)}{s} \mathrm{d}s = \infty.$$

Let $\mu \in (0,1)$ be a given constant. There exists $T > a$ such that (4.27) holds for $t \ge T$ and

$$2\delta_L(t) \le \mu \ \text{ for } t \ge T, \quad \frac{2K}{L^{\gamma-1}(T)}. \tag{4.28}$$

Define Ξ to be the set of positive nondecreasing functions $\xi(t)$ on $[T, \infty)$ satisfying $1 \le \xi(t) \le L(t)/L(T)$ for $t \ge T$. It is clear that Ξ is a closed convex subset of the locally convex space $C[T, \infty)$. For $\xi \in \Xi$ put

$$p_\xi(t) = p(t)\xi^{\gamma-1}(t), \ \ Q_\xi(t) = t \int_t^\infty p_\xi(s)\,\mathrm{d}s,$$

and consider the family of ordinary linear differential equations

$$x'' + p_\xi(t)x = 0, \quad \xi \in \Xi. \tag{4.29}$$

For each $\xi \in \Xi$ we have by (4.27) and (4.28)

$$Q_\xi(t) \le t \int_t^\infty p(s) \left(\frac{L(s)}{L(T)} \right)^{\gamma-1} \mathrm{d}s \le \frac{K\delta_L(t)}{L^{\gamma-1}(T)} \le \frac{\delta_L(t)}{2} = \varphi(t), \tag{4.30}$$

$t \geq T$, and so from Lemma 4.2, we see that for each $\xi \in \Xi$ equation (4.29) has a $\mathcal{SV}$ solution x_ξ, which is increasing and expressed in the form

$$x_\xi(t) = \exp\left\{\int_T^t \frac{v_\xi(s) + Q_\xi(s)}{s} \mathrm{d}s\right\}, \tag{4.31}$$

$t \geq T$, where v_ξ is a unique solution of the integral equation

$$v_\xi(t) = t\int_t^\infty \left(\frac{v_\xi(s) + Q_\xi(s)}{s}\right)^2 \mathrm{d}s,$$

$t \geq T$, and

$$v_\xi(t) \leq \delta_L^2(t) \leq \frac{\delta_L(t)}{2}. \tag{4.32}$$

Using (4.30) and (4.32) in (4.31), we conclude that

$$1 \leq x_\xi(t) \leq \exp\left\{\int_T^t \frac{\delta_L(s)}{s}\,\mathrm{d}s\right\} = \frac{L(t)}{L(T)},$$

$t \geq T$. Let us now define the mapping $\mathcal{T} : \Xi \to C([T, \infty)$ by

$$\mathcal{T}\xi(t) = x_\xi(t),$$

$t \geq T$. It is immediate that $\mathcal{T}\Xi \subset \Xi$. Further, relative compactness of $\mathcal{T}\Xi$ follows from the Arzela-Ascoli lemma. To show that $\mathcal{T}$ is continuous is a little bit laborious, and we just note here that the Lebesgue dominated convergence theorem plays an important role. For details see [85]. Thus, all the hypotheses of the Schauder-Tychonoff fixed point theorem are fulfilled, and so there exists $\xi \in \Xi$ such that $\xi(t) = \mathcal{T}\xi(t) = x_\xi(t)$ for $t \geq T$. Since $x_\xi(t)$ is a solution of linear differential equation (4.29), we have for $t \geq T$ that

$$0 = x_\xi''(t) + p_\xi(t)x_\xi(t) = \xi''(t) + p(t)\xi^{\gamma-1}(t)\xi(t) = \xi''(t) + p(t)\xi^\gamma(t),$$

which implies that ξ is a solution of equation (4.24). It is obvious that $\xi \in \mathcal{SV}$. □

Example 1 in [85] suggests that super-linear Emden-Fowler equations may have both trivial and nontrivial $\mathcal{SV}$ solutions at the same time. It is therefore important to establish a means by which one can distinguish nontrivial $\mathcal{SV}$ solutions from trivial ones for a given Emden-Fowler equation. Under the stronger assumption than (4.27), one can determine the exact asymptotic behavior of any nontrivial $\mathcal{SV}$ solution of super-linear Emden-Fowler equations as the following theorem shows.

Theorem 4.8. *Let $\gamma > 1$ and (4.26) hold. Suppose that $p \in \mathcal{RV}(-\nu)$ for some $\nu > 1$ and that*

$$Q_L(t) \sim t\frac{L'(t)}{L(t)} = \delta_L(t)$$

as $t \to \infty$, where δ_L is required to be $\mathcal{SV}$ function. Then equation (4.24) *may possess a $\mathcal{SV}$ solution y only if $\nu = 2$, and in this case any nontrivial $\mathcal{SV}$ solution of equation* (4.24) *has one and the same asymptotic behavior*

$$y(t) \sim L(t) \tag{4.33}$$

as $t \to \infty$.

Proof. Since $p \in \mathcal{RV}(-\nu)$, we have that $p(t) = t^{-\nu} L_p(t)$, $L_p \in \mathcal{SV}$, and application of the Karamata integration theorem gives

$$Q_L(t) = t \int_t^\infty s^{-\nu} L_p(s) L^{\gamma-1}(s)\,\mathrm{d}s \sim \frac{t^{2-\nu}}{\nu-1} L_p(t) L^{\gamma-1}(t) \tag{4.34}$$

as $t \to \infty$. Thus, due to the assumption that $Q_L(t) \sim \delta_L(t) \in \mathcal{SV}$ as $t \to \infty$, from (4.34) we conclude that $\nu = 2$ and

$$L_p(t) \sim t \frac{L'(t)}{L^\gamma(t)}$$

as $t \to \infty$. Suppose that (4.24) has a nontrivial $\mathcal{SV}$ solution y. Integrating (4.24) from t to ∞ and using the Karamata integration theorem, we have

$$y'(t) = \int_t^\infty p(s) y^\gamma(s)\,\mathrm{d}s = \int_t^\infty \frac{L_p(s)}{s^2} y^\gamma(s)\,\mathrm{d}s \sim \frac{L_p(t)}{t} y^\gamma(t) \sim \frac{L'(t)}{L^\gamma(t)} y^\gamma(t)$$

as $t \to \infty$, which implies that

$$\frac{y'(t)}{y^\gamma(t)} \sim \frac{L'(t)}{L^\gamma(t)} \tag{4.35}$$

as $t \to \infty$. Integrating (4.35) over $[t, \infty)$ and noting that $y(t) \to \infty$ and $L(t) \to \infty$ as $t \to \infty$, we obtain

$$\frac{y^{1-\gamma}(t)}{\gamma - 1} \sim \frac{L^{1-\gamma}(t)}{\gamma - 1}$$

as $\to \infty$, which immediately yields (4.33). □

4.3.4 Intermediate generalized $\mathcal{RV}$ solutions of second order sub-half-linear equations

In this section which is based on the paper [91] by Kusano, Manojlović, and Milošević we want to concisely describe how the approach based on the approximation of certain Emden-Fowler equation in an integral form by means of an asymptotic relation and on the use of the fixed point theorem can be combined with the concept of general regularly varying functions.

Consider the equation

$$(r(t)\Phi_\alpha(y'))' + p(t)\Phi_\beta(y) = 0, \tag{4.36}$$

where $\alpha > \beta > 0$ (which is the sub-half-linearity condition), r, p are continuous functions, and r satisfies $\int_a^\infty r^{-\frac{1}{\alpha}}(t)\,\mathrm{d}t < \infty$. Note that a similar analysis in the case where this integral diverges was made by Jaroš, Kusano, Manojlović in [64]. Denote

$$R(t) = \int_t^\infty r^{-\frac{1}{\alpha}}(s)\,\mathrm{d}s.$$

We have the following conventional classification of positive solutions :

(i) $\lim_{t\to\infty} y(t) = \text{const} > 0$,

(ii) $\lim_{t\to\infty} y(t) = 0$, $\lim_{t\to\infty} y(t)/R(t) = \infty$,

(iii) $\lim_{t\to\infty} y(t)/R(t) = \text{const} > 0$.

Solutions of type (ii) are called intermediate solutions. Sharp conditions for the existence of such solutions was obtained by Kamo and Usami in [72].

Let y be an intermediate solution of (4.36) on $[a, \infty)$. Then $\int_a^\infty p(s)y^\beta(s)\,\mathrm{d}s = \infty$ and

$$y(t) = \int_t^\infty \frac{1}{r^{\frac{1}{\alpha}}}\left(r(a)(-y'(s))^\alpha + \int_a^s p(\tau)y^\beta(\tau)\,\mathrm{d}\tau\right)^{\frac{1}{\alpha}} \mathrm{d}s, \tag{4.37}$$

$t \ge a$. It follows therefore that y satisfies the integral asymptotic relation

$$y(t) \sim \int_t^\infty \frac{1}{r^{\frac{1}{\alpha}}}\left(\int_a^s p(\tau)y^\beta(\tau)\,\mathrm{d}\tau\right)^{\frac{1}{\alpha}} \mathrm{d}s, \tag{4.38}$$

as $t \to \infty$, which is regarded as an "approximation" of (4.37) at infinity. The proofs of the main results are essentially based on the fact that a thorough knowledge of the existence and asymptotic behavior of generalized $\mathcal{RV}$ solutions of (4.38) can be acquired. As a matter of fact, the "only if" part of the below presented theorem is an immediate consequence of manipulation of (4.38) by means of regular variation. The "if" part is proved by solving the integral equation

$$y(t) = \int_t^\infty \frac{1}{r^{\frac{1}{\alpha}}}\left(\int_a^s p(\tau)y^\beta(\tau)\,\mathrm{d}\tau\right)^{\frac{1}{\alpha}} \mathrm{d}s,$$

with the help of fixed point technique, the essence of which is based on detecting fixed points of the integral operator

$$\mathcal{F}y(t) = \int_t^\infty \frac{1}{r^{\frac{1}{\alpha}}}\left(\int_a^s p(\tau)y^\beta(\tau)\,\mathrm{d}\tau\right)^{\frac{1}{\alpha}} \mathrm{d}s,$$

lying in the set

$$\mathcal{X} = \{y \in C[a, \infty) : mx(t) \le y(t) \le Mx(t), t \ge a\},$$

m, M being a suitable numbers, and

$$x(t) = \left(\frac{R^{\alpha+1}(t)r^{\frac{1}{\alpha}}(t)p(t)}{\alpha(-\rho)^\alpha(\rho+1)}\right)^{\frac{1}{\alpha-\beta}},$$

where ρ is given below in the theorem. The operator $\mathcal{F}$ is continuous self-map on $\mathcal{X}$ and sends it into a relatively compact subset of $C[a, \infty)$. To show that the obtained solution y is indeed $\mathcal{RV}$ with respect to $1/R$, we apply the generalized L'Hospital rule which yields

$$y(t) \sim \int_t^\infty \frac{1}{r^{\frac{1}{\alpha}}} \left(\int_a^s p(\tau) x^\beta(\tau) \, \mathrm{d}\tau \right)^{\frac{1}{\alpha}} \mathrm{d}s$$

as $t \to \infty$.

Thus we have briefly described the proof of the following theorem.

Theorem 4.9. *Let $r \in \mathcal{RV}_{1/R}(\eta)$ and $p \in \mathcal{RV}_{1/R}(\sigma)$. Equation* (4.36) *has intermediate solutions $y \in \mathcal{RV}_{1/R}(\rho)$ with $\rho \in (-1, 0)$ if and only if*

$$\beta - \frac{\eta}{\alpha} + 1 < \sigma < \alpha - \frac{\eta}{\alpha} + 1, \tag{4.39}$$

in which case ρ is given by

$$\rho = \frac{\sigma - \alpha - 1 + \frac{\eta}{\alpha}}{\alpha - \beta}$$

and asymptotic behavior of any such solution y is governed by the unique formula

$$y(t) \sim \left(\frac{R^{\alpha+1}(t) r^{\frac{1}{\alpha}}(t) p(t)}{\alpha(-\rho)^\alpha(\rho + 1)} \right)^{\frac{1}{\alpha-\beta}}$$

as $t \to \infty$.

Note that the cases where σ is equal to the border values in (4.39) are also discussed in [91] and lead to the existence of nontrivial $\mathcal{SV}_{1/R}$ and $\mathcal{RV}_{1/R}(-1)$ solutions.

4.3.5 Strongly monotone solutions of coupled systems

In the first part of this section we present the results concerning the so-called strongly decreasing solutions of the coupled system

$$\begin{cases} (p(t)\Phi_\alpha(x'))' = \varphi(t)\Phi_\lambda(y), \\ (q(t)\Phi_\beta(y'))' = \psi(t)\Phi_\mu(x), \end{cases} \tag{4.40}$$

which were established by Matucci, Řehák in [123]. We do not give the proof since its main ideas can be extracted from the proof of subsequent Theorem 4.15, which deals with a more general case. Note that originally, there are some differences between the proofs. Rather we focus on comments and applications. In the second part, we mention the work of Jaroš, Kusano [62] in which the coupled system in a more special form is considered; the approach used there is based on similar ideas as in the previous subsections. We present a result which deals with mixed strongly monotone solutions.

In (4.40) we assume that $\alpha, \beta, \lambda, \mu$ are positive constants, and p, q, φ, ψ are positive continuous functions defined on $[a, \infty)$, $a \geq 0$. Further we suppose that system (4.40) is *subhomogeneous* (at ∞), i.e.,

$$\alpha\beta > \lambda\mu.$$

In contrast to the most of related works, we do not pose in general any condition on divergence or convergence of the integrals

$$P = \int_a^\infty p^{-\frac{1}{\alpha}}(s)\,\mathrm{d}s, \quad Q = \int_a^\infty q^{-\frac{1}{\beta}}(s)\,\mathrm{d}s, \tag{4.41}$$

and we do not explicitly distinguish among particular cases. In fact, all possible cases (including the mixed ones) are covered by the results. If (x, y) is a solution of (4.40), then by *quasiderivatives* denoted as $x^{[1]}, y^{[1]}$, we mean

$$x^{[1]} = p\,\Phi_\alpha(x'), \quad y^{[1]} = q\,\Phi_\beta(y').$$

Let $\mathcal{DS}$ denote the set of all positive decreasing solutions of (4.40), i.e., all solutions whose components are both eventually positive and decreasing. Note that, due to the sign conditions on the coefficients, any solution of (4.40) has necessarily both components eventually of one sign and monotone. If $P = Q = \infty$, then $\lim_{t\to\infty} x^{[1]}(t) = \lim_{t\to\infty} y^{[1]}(t) = 0$ for any $(x, y) \in \mathcal{DS}$. Indeed, $-x^{[1]}$ is eventually positive decreasing. If $\lim_{t\to\infty} -x^{[1]}(t) = c > 0$, then $p(t)(-x'(t))^\alpha \geq c$ or $-x'(t) \geq c^{\frac{1}{\alpha}} p^{-\frac{1}{\alpha}}(t)$, $t \geq t_0$ with some $t_0 \geq a$. Integrating the latter inequality, we get $x(t) \leq x(t_0) - c^{\frac{1}{\alpha}} \int_{t_0}^t p^{-\frac{1}{\alpha}}(s)\,\mathrm{d}s \to -\infty$ as $t \to \infty$, a contradiction. Similarly we prove $\lim_{t\to\infty} y^{[1]}(t) = 0$. However, in general, for positive decreasing solutions, the limits of quasiderivatives do not need to be zero. Observe that any positive decreasing solution have both the components and their quasiderivatives tending to nonnegative resp. nonpositive numbers. Among all these solutions we are interested in the so called *strongly decreasing solutions*, which we denote as

$$\mathcal{SDS} = \left\{(x, y) \in \mathcal{DS} : \lim_{t\to\infty} x(t) = \lim_{t\to\infty} y(t) = \lim_{t\to\infty} x^{[1]}(t) = \lim_{t\to\infty} y^{[1]}(t) = 0\right\};$$

notice that other types of $\mathcal{DS}$ solutions are somehow easier.

The integral expressions below play important roles in existence results for strongly decreasing solutions:

$$I_1(t) := \int_t^\infty \left(\frac{1}{p(u)} \int_u^\infty \varphi(s) \left(\int_s^\infty \left(\frac{1}{q(r)} \int_r^\infty \psi(\tau)\,\mathrm{d}\tau \right)^{\frac{1}{\beta}} \mathrm{d}r \right)^{\lambda} \mathrm{d}s \right)^{\frac{1}{\alpha}} \mathrm{d}u,$$

$$I_2(t) := \int_t^\infty \left(\frac{1}{q(u)} \int_u^\infty \psi(s) \left(\int_s^\infty \left(\frac{1}{p(r)} \int_r^\infty \varphi(\tau)\,\mathrm{d}\tau \right)^{\frac{1}{\alpha}} \mathrm{d}r \right)^{\mu} \mathrm{d}s \right)^{\frac{1}{\beta}} \mathrm{d}u,$$

$$I_3(t) := \int_t^\infty \varphi(u) \left(\int_u^\infty \left(\frac{1}{q(s)} \int_s^\infty \psi(r) \left(\int_r^\infty \frac{1}{p^{\frac{1}{\alpha}}(\tau)}\,\mathrm{d}\tau \right)^{\mu} \mathrm{d}r \right)^{\frac{1}{\beta}} \mathrm{d}s \right)^{\lambda} \mathrm{d}u,$$

$$I_4(t) := \int_t^\infty \psi(u) \left(\int_u^\infty \left(\frac{1}{p(s)} \int_s^\infty \varphi(r) \left(\int_r^\infty \frac{1}{q^{\frac{1}{\beta}}(\tau)}\,\mathrm{d}\tau \right)^{\lambda} \mathrm{d}r \right)^{\frac{1}{\alpha}} \mathrm{d}s \right)^{\mu} \mathrm{d}u.$$

The following existence result holds. Its proof can be found in the paper [123] by Matucci, Řehák; the Schauder-Tychonoff fixed point theorem is the main tool.

Theorem 4.10. *If $I_i(a) < \infty$ for at least one index $i \in \{1, 2, 3, 4\}$, then $\mathcal{SDS} \neq \emptyset$.*

Next we describe the so-called *reciprocity principle*. This tool is very useful in the proof of the existence result, but also in the proof of the main result devoted to exact asymptotic behavior. In fact, it enables us to extend the existence and asymptotic results made under certain setting to "new" situations. Let $(x, y) \in \mathcal{DS}$. Set $u = -x^{[1]}$, $v = -y^{[1]}$. Then (u, v) is an eventually positive decreasing solution of the *reciprocal system*

$$\begin{cases} \left(\dfrac{1}{\varphi^{\frac{1}{\lambda}}(t)} \Phi_{\frac{1}{\lambda}}(u') \right)' = \dfrac{1}{q^{\frac{1}{\beta}}} \Phi_{\frac{1}{\beta}}(v), \\ \left(\dfrac{1}{\psi^{\frac{1}{\mu}}(t)} \Phi_{\frac{1}{\mu}}(v') \right)' = \dfrac{1}{p^{\frac{1}{\alpha}}} \Phi_{\frac{1}{\alpha}}(u). \end{cases} \tag{4.42}$$

Observe that (4.42) has the same structure as (4.40), and is subhomogeneous. Indeed, $1/(\lambda\mu) > 1/(\alpha\beta)$. Moreover, the quasiderivatives of u and v, i.e., $\varphi^{-\frac{1}{\lambda}} \Phi_{\frac{1}{\lambda}}(u')$ and $\psi^{-\frac{1}{\mu}} \Phi_{\frac{1}{\mu}}(v')$, are equal to $-y$ and $-x$, respectively. Conversely, if (u, v) is an eventually positive decreasing solution of (4.42) and we set $x = -\psi^{-\frac{1}{\mu}} \Phi_{\frac{1}{\mu}}(v')$ and $y = -\varphi^{-\frac{1}{\lambda}} \Phi_{\frac{1}{\lambda}}(u')$, then $(x, y) \in \mathcal{DS}$ and $x^{[1]} = -u$, $y^{[1]} = -v$. Hence, with the use of these relations, it holds

$$\begin{gathered} (x, y) \text{ is a strongly decreasing solution of (4.40)} \\ \Updownarrow \\ (u, v) \text{ is a strongly decreasing solution of (4.42).} \end{gathered} \tag{4.43}$$

It is easy to see that the roles which are played by the integrals I_1 and I_2 for system (4.40) are played by the integrals I_3 and I_4, respectively, for system (4.42).

From now on we assume

$$p \in \mathcal{RV}(\gamma),\ q \in \mathcal{RV}(\delta),\ \varphi \in \mathcal{RV}(\sigma),\ \psi \in \mathcal{RV}(\varrho).$$

We set

$$\Lambda = \frac{1}{\alpha\beta - \lambda\mu}$$

and

$$\begin{aligned} \nu &= \Lambda\Big(\beta(\alpha - \gamma + 1 + \sigma) + \lambda(\beta - \delta + 1 + \varrho)\Big), \\ \omega &= \Lambda\Big(\alpha(\beta - \delta + 1 + \varrho) + \mu(\alpha - \gamma + 1 + \sigma)\Big). \end{aligned} \tag{4.44}$$

Further we denote

$$\nu^{[1]} = (\nu - 1)\alpha + \gamma,\ \ \omega^{[1]} = (\omega - 1)\beta + \delta. \tag{4.45}$$

Theorem 4.11. *Assume*

$$\nu < 0,\ \omega < 0,\ \nu^{[1]} < 0,\ \omega^{[1]} < 0. \tag{4.46}$$

Then $\mathcal{SDS} \neq \emptyset$. Further, for every $(x, y) \in \mathcal{SDS}$ there hold $(x, y) \in \mathcal{RV}(\nu) \times \mathcal{RV}(\omega)$ and

$$x(t) \sim \left(K_1 K_2^{\frac{\lambda}{\alpha}}\right)^{\alpha\beta\Lambda} t^{\nu} L_1(t), \quad y(t) \sim \left(K_2 K_1^{\frac{\mu}{\beta}}\right)^{\alpha\beta\Lambda} t^{\omega} L_2(t) \tag{4.47}$$

as $t \to \infty$, where

$$K_1 = \frac{-1}{\nu\,(-\nu^{[1]})^{\frac{1}{\alpha}}}, \quad K_2 = \frac{-1}{\omega\,(-\omega^{[1]})^{\frac{1}{\beta}}}, \tag{4.48}$$

and

$$L_1 = \left(\frac{L_\varphi^\beta L_\psi^\lambda}{L_p^\beta L_q^\lambda}\right)^{\Lambda} \in \mathcal{SV}, \quad L_2 = \left(\frac{L_\varphi^\mu L_\psi^\alpha}{L_p^\mu L_q^\alpha}\right)^{\Lambda} \in \mathcal{SV}.$$

Remark 4.4. (i) Under the assumptions of Theorem 4.11, the quasiderivatives of a strongly decreasing solution (x, y) of (4.40) satisfy

$$\begin{aligned} -x^{[1]}(t) &\sim \frac{\left(K_2 K_1^{\frac{\mu}{\beta}}\right)^{\alpha\beta\Lambda\lambda}}{-\nu^{[1]}} t^{\nu^{[1]}} L_\varphi(t) L_2^{\lambda}(t) \in \mathcal{RV}\left(\nu^{[1]}\right) \\ -y^{[1]}(t) &\sim \frac{\left(K_1 K_2^{\frac{\lambda}{\alpha}}\right)^{\alpha\beta\Lambda\mu}}{-\omega^{[1]}} t^{\omega^{[1]}} L_\psi(t) L_1^{\mu}(t) \in \mathcal{RV}\left(\omega^{[1]}\right). \end{aligned}$$

(ii) The asymptotic formula (4.47) can alternatively be written in terms of the coefficients,

$$x(t) \sim C_1 \left(\frac{t^{\alpha+1}\varphi(t)}{p(t)}\right)^{\beta/(\alpha\beta-\lambda\mu)} \left(\frac{t^{\beta+1}\psi(t)}{q(t)}\right)^{\lambda/(\alpha\beta-\lambda\mu)},$$
$$y(t) \sim C_2 \left(\frac{t^{\beta+1}\psi(t)}{q(t)}\right)^{\alpha/(\alpha\beta-\lambda\mu)} \left(\frac{t^{\alpha+1}\varphi(t)}{p(t)}\right)^{\mu/(\alpha\beta-\lambda\mu)}$$

as $t \to \infty$, where $C_1 = (K_1 K_2^{\lambda/\alpha})^{\alpha\beta/(\alpha\beta-\lambda\mu)}$ and $C_2 = (K_2 K_1^{\mu/\beta})^{\alpha\beta/(\alpha\beta-\lambda\mu)}$.

(iii) If the components of a (decreasing) regularly varying solution to (4.40) are eventually convex, then they are both normalized regularly varying. Note that the eventual convexity of decreasing solutions of (4.40) can be guaranteed, e.g., by $p(t) = q(t) = 1$.

(iv) The assumptions of Theorem 4.11 can be replaced by any of the following inequalities

$$\varrho + 1 < \min\left\{0, \delta - \beta, \delta - \beta - \frac{\beta}{\lambda}(\sigma + 1), \delta - \beta - \frac{\beta}{\lambda}(\sigma + 1 + \alpha - \gamma)\right\} \tag{4.49}$$

or

$$\sigma + 1 < \min\left\{0, \gamma - \alpha, \gamma - \alpha - \frac{\alpha}{\mu}(\varrho + 1), \gamma - \alpha - \frac{\alpha}{\mu}(\varrho + 1 + \beta - \delta)\right\} \tag{4.50}$$

or

$$\begin{aligned} \alpha - \gamma < \min\Big\{0, -\frac{\alpha}{\mu}(\varrho + 1), -\frac{\alpha}{\mu}(\varrho + 1 + \beta - \delta),& \\ -\frac{\alpha}{\mu}(\varrho + 1 + \beta - \delta) - \frac{\alpha\beta}{\lambda\mu}(\sigma + 1)\Big\}& \end{aligned} \tag{4.51}$$

or

$$\begin{aligned} \beta - \delta < \min\Big\{0, -\frac{\beta}{\lambda}(\sigma + 1), -\frac{\beta}{\lambda}(\sigma + 1 + \alpha - \gamma),& \\ -\frac{\beta}{\lambda}(\sigma + 1 + \alpha - \gamma) - \frac{\alpha\beta}{\lambda\mu}(\varrho + 1)\Big\},& \end{aligned} \tag{4.52}$$

and the statement remains valid. Moreover, the following equivalence among the sufficient conditions hold:

$$(4.46) \quad \Leftrightarrow \quad (4.49) \text{ or } (4.50) \text{ or } (4.51) \text{ or } (4.52). \tag{4.53}$$

As an example of typical setting, assume $P = Q = \infty$. Then necessarily $\alpha \geq \gamma$ and $\beta \geq \delta$; the sufficient condition (4.46) reduces to

$$\nu < 0, \omega < 0. \tag{4.54}$$

Indeed, $\nu^{[1]} = \nu\alpha - \alpha + \gamma \le \nu\gamma < 0$. Similarly we obtain $\omega^{[1]} < 0$.

Another example of a typical situation is when

$$\int_a^\infty \varphi(t)\,\mathrm{d}t = \int_a^\infty \psi(t)\,\mathrm{d}t = \infty$$

is assumed. This condition implies $\sigma + 1 \ge 0$, $\varrho + 1 \ge 0$, and (4.46) reduces to $\nu^{[1]} < 0, \omega^{[1]} < 0$. It can be also quite easily observed how the "mixed" cases, for instance,

$$\int_a^\infty p^{-\frac{1}{\alpha}}(s)\,\mathrm{d}s = \infty, \int_a^\infty q^{-\frac{1}{\beta}}(s)\,\mathrm{d}s < \infty,$$

or

$$\int_a^\infty \varphi(t)\,\mathrm{d}t = \infty, \int_a^\infty \psi(t)\,\mathrm{d}t < \infty$$

can be covered by the above results.

The main theorem can be applied also in some special situations where the coefficients of (4.40) are not regularly varying; we use a change of the independent variable. Let $s = \zeta(t)$, where ζ is a differentiable function such that $\zeta'(t) \neq 0$ on $[a, \infty)$. Set $(w, z)(s) = (x, y)(\zeta^{-1}(s))$, ζ^{-1} being the inverse of ζ. Since $\mathrm{d}/\mathrm{d}t = \zeta'(t)\mathrm{d}/\mathrm{d}s$, system (4.40) is transformed into the system

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}s}\left(\hat{p}(s)\Phi_\alpha\left(\dfrac{\mathrm{d}w}{\mathrm{d}s}\right)\right) = \hat{\varphi}(s)\Phi_\lambda(z), \\ \dfrac{\mathrm{d}}{\mathrm{d}s}\left(\hat{q}(s)\Phi_\beta\left(\dfrac{\mathrm{d}z}{\mathrm{d}s}\right)\right) = \hat{\psi}(s)\Phi_\mu(w), \end{cases} \tag{4.55}$$

where

$$\hat{p} := (p \circ \zeta^{-1}) \cdot \Phi_\alpha(\zeta' \circ \zeta^{-1}), \;\; \hat{q} := (q \circ \zeta^{-1}) \cdot \Phi_\beta(\zeta' \circ \zeta^{-1}),$$
$$\hat{\varphi} := \frac{\varphi \circ \zeta^{-1}}{\zeta' \circ \zeta^{-1}}, \;\; \hat{\psi} := \frac{\psi \circ \zeta^{-1}}{\zeta' \circ \zeta^{-1}}.$$

Clearly, (4.55) is of the form (4.40) and is subhomogeneous. If ζ is unbounded with $\zeta' > 0$ and such that $\hat{p}, \hat{q}, \hat{\varphi}, \hat{\psi} \in \bigcup_{\vartheta\in\mathbb{R}} \mathcal{RV}(\vartheta)$, then the results can be applied to (4.55). To illustrate a possible application, take, for example,

$$p(t) = e^{\gamma t}h_1(t),\; q(t) = e^{\delta t}h_2(t),\; \varphi(t) = e^{\sigma t}h_3(t),\; \psi(t) = e^{\varrho t}h_4(t),$$

where $\gamma, \delta, \sigma, \varrho \in \mathbb{R}$ and $h_i \in \bigcup_{\vartheta\in\mathbb{R}} \mathcal{RV}(\vartheta)$, $i = 1, 2, 3, 4$. In such a case we can set $\zeta(t) = e^t$. Thus $t = \ln s$ and $[a, \infty)$ is transformed into another right half-line. We then get

$$\hat{p}(s) = s^{\gamma+\alpha}H_1(s) \in \mathcal{RV}(\gamma + \alpha), \;\; \text{where } H_1 := h_1 \circ \ln,$$

since $H_1 \in \mathcal{RV}(\gamma_1 \cdot 0) = \mathcal{RV}(0) = \mathcal{SV}$, γ_1 being the index of regular variation of h_1. Similarly,

$$\hat{q} \in \mathcal{RV}(\delta + \beta), \;\; \hat{\varphi} \in \mathcal{RV}(\sigma - 1), \;\; \hat{\psi} \in \mathcal{RV}(\varrho - 1).$$

Consequently, Theorem 4.11 can directly be applied to system (4.55), and hereby to the original system through the transformation.

We conclude this part by the application of Theorem 4.11 to the widely studied 4th-order equation

$$x'''' = \psi(t)\Phi_\mu(x). \tag{4.56}$$

The below mentioned observations can be easily extended to some more general 4th-order equations, but for comparison purposes we take the form (4.56), which appears quite frequently in the literature. We assume $\psi \in \mathcal{RV}(\varrho)$ and $\mu > 0$. This equation is equivalent to system (4.40), where we set $\alpha = \beta = \lambda = 1$ and $p(t) = q(t) = \varphi(t) = 1$. Then $\gamma = \delta = \sigma = 0$, $L_p(t) = L_q(t) = L_\varphi(t) = 1$, and $\Lambda = 1/(1-\mu)$. The subhomogeneity assumption reads as $\mu < 1$. Further,

$$\nu = \frac{\varrho+4}{1-\mu}, \quad \omega = \frac{\varrho+2+2\mu}{1-\mu}, \quad \nu^{[1]} = \frac{\varrho+3+\mu}{1-\mu}, \quad \omega^{[1]} = \frac{\varrho+1+3\mu}{1-\mu}.$$

A strongly decreasing solution x of (4.56) is a positive solution such that

$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty} x'(t) = \lim_{t\to\infty} x''(t) = \lim_{t\to\infty} x'''(t) = 0.$$

It is easy to see that in order (4.46) to be fulfilled, it is sufficient to take $\varrho < -4$. Alternatively, we can check how (4.49) is verified (this is the only one among conditions (4.49), (4.50), (4.51), (4.52), that can be satisfied in this situation). Thus, under the assumptions $0 < \mu < 1$ and $\varrho < -4$, Theorem 4.11 assures that (4.56) possesses a strongly decreasing solution and for any such a solution x it holds

$$x(t) \sim \left(\frac{(1-\mu)^4 t^{\varrho+4} L_\psi(t)}{\prod_{i=1}^4 [\varrho + i + (4-i)\mu]} \right)^{\frac{1}{1-\mu}}$$

as $t \to \infty$.

A special case of (4.40) in the form

$$x'' = \varphi(t) y^\lambda, \quad y'' = \psi(t) x^\mu, \tag{4.57}$$

$\varphi(t) > 0, \psi(t) > 0$, $\lambda, \mu > 0$, $\lambda\mu < 1$, was investigated in [62] by Jaroš and Kusano. More precisely, strongly monotone solutions were studied there. Since the approach is based on similar ideas to those described in Subsections 4.3.2–4.3.4 (approximation by means of a suitable integral asymptotic relation, a-priori bounds, and the Schauder fixed point theorem) we present one selected result without proof. It concerns existence and asymptotic behavior of mixed strongly monotone solutions to (4.57), i.e., the positive solutions where one of the components is strongly decreasing, while the other is strongly increasing.

Theorem 4.12. *Suppose that $\varphi \in \mathcal{RV}(\sigma)$, $\psi \in \mathcal{RV}(\varrho)$. System* (4.57) *possesses regularly varying solutions (x, y) such that*

$$x \in \mathcal{RV}(\nu), \ \ y \in \mathcal{RV}(\omega), \ \ \nu < 1, \ \ \omega > 1,$$

if and only if

$$\sigma + 2 + \lambda(\varrho + 2) < 0, \quad \mu(\lambda + \sigma + 2) + \varrho + 1 > 0,$$

in which case ν and ω are given by

$$\nu = \frac{\sigma + 2 + \lambda(\varrho + 2)}{1 - \lambda\mu}, \quad \omega = \frac{\mu(\sigma + 2) + \varrho + 2}{1 - \lambda\mu},$$

and the asymptotic behavior of any such solution is governed by the formulas

$$x(t) \sim \left(\frac{t^{2(\lambda+1)}\varphi(t)\psi^{\lambda}(t)}{\nu(\nu-1)[\omega(\omega-1)]^{\lambda}} \right)^{\frac{1}{1-\lambda\mu}}, \quad y(t) \sim \left(\frac{t^{2(\mu+1)}\varphi^{\mu}(t)\psi(t)}{[\nu(\nu-1)]^{\mu}\omega(\omega-1)} \right)^{\frac{1}{1-\lambda\mu}}$$

as $t \to \infty$.

4.3.6 Overall structure of $\mathcal{RV}$ solutions to odd-order equations

In this section we present a description of overall structure of $\mathcal{RV}$ solutions to the equation

$$y^{2n+1} + p(t)\Phi_{\gamma}(y) = 0, \tag{4.58}$$

where $0 < \gamma < 1$ and $p : [a, \infty) \to (0, \infty)$ is a continuous function. Such a description is the result of considerations made by Kusano and Manojlović in [89]. Some of the ideas in the proofs are similar to those used e.g. in Subsection 4.3.3. In particular, here again an integral asymptotic relation is investigated, which can be seen as an approximation of an integral form of equation (4.58); this time the relation reads as

$$y(t) \sim \int_a^t \frac{(t-s)^{k-1}}{(k-1)!} \int_s^{\infty} \frac{(r-s)^{2n-k}}{(2n-k)!} p(\tau) y^{\gamma}(\tau) \mathrm{d}\tau \, \mathrm{d}s,$$

k is assumed to be even integer such that $2 \le k \le 2n$. In addition, the tools like the Schauder-Tychonoff fixed point theorem and the Karamata integration theorem again play important roles in the proofs. In classifying eventually positive solutions of (4.58), the well-known Kiguradze lemma is utilized.

Assume that $p \in \mathcal{RV}(\sigma)$. Denote with $\mathcal{RS}$ the set of all $\mathcal{RV}$ solutions of (4.58), and define the subsets

$$\begin{aligned} \mathcal{RS}(\varrho) &= \mathcal{RS} \cap \mathcal{RV}(\varrho), \\ \text{tr-}\mathcal{RS}(\varrho) &= \mathcal{RS} \cap \text{tr-}\mathcal{RV}(\varrho), \\ \text{ntr-}\mathcal{RS}(\varrho) &= \mathcal{RS} \cap \text{ntr-}\mathcal{RV}(\varrho). \end{aligned}$$

Using notation $\gamma_m = m(1-\gamma) - 2n - 1$, $m \in \{0, 1, 2, \ldots, 2n\}$, to make the full analysis we separately consider the case $\sigma < \gamma_0 = -2n - 1$ together with the central cases

$$\sigma \in (\gamma_0, \gamma_1) \cup (\gamma_2, \gamma_3) \cup \cdots \cup (\gamma_{2n-2}, \gamma_{2n-1})$$

or

$$\sigma \in (\gamma_1, \gamma_2) \cup (\gamma_3, \gamma_4) \cup \cdots \cup (\gamma_{2n-1}, \gamma_{2n}),$$

and the border cases

$$\sigma \in \{\gamma_0, \gamma_2, \dots, \gamma_{2n}\} \text{ or } \sigma \in \{\gamma_1, \gamma_3, \dots, \gamma_{2n-1}\}.$$

Denote

$$Q_j = \int_a^\infty t^{2n-j(1-\gamma)} p(s)\,\mathrm{d}s,$$

$j = 0, 1, \dots, 2n$. The structure of $\mathcal{RV}$ solutions to equation (4.58) reads then as follows:

(i) If $\sigma < \gamma_0$, then

$$\mathcal{RS} = \bigcup_{j=0}^{2n} \text{tr-}\mathcal{RS}(j) \cup \mathcal{RS}\left(\frac{\sigma + 2n + 1}{1 - \gamma}\right).$$

(ii) If $\sigma = \gamma_0$ and $Q_0 < \infty$, then

$$\mathcal{RS} = \bigcup_{j=0}^{2n} \text{tr-}\mathcal{RS}(j) \cup \text{ntr-}\mathcal{RS}(0).$$

(iii) If $\sigma = \gamma_0$ and $Q_0 = \infty$, then

$$\mathcal{RS} = \bigcup_{j=1}^{2n} \text{tr-}\mathcal{RS}(j).$$

(iv) If $\sigma = \gamma_m$ for some $m \in \{1, 3, \dots, 2n-1\}$ and $Q_m < \infty$, then

$$\mathcal{RS} = \bigcup_{j=m}^{2n} \text{tr-}\mathcal{RS}(j).$$

(v) If $\sigma = \gamma_m$ for some $m \in \{1, 3, \dots, 2n-1\}$ and $Q_m = \infty$, then

$$\mathcal{RS} = \bigcup_{j=m+1}^{2n} \text{tr-}\mathcal{RS}(j) \cup \text{ntr-}\mathcal{RS}(m).$$

(vi) If $\sigma \in (\gamma_{m-1}, \gamma_m)$ for some $m \in \{2, 4, \dots, 2n-2\}$, then

$$\mathcal{RS} = \bigcup_{j=m}^{2n} \text{tr-}\mathcal{RS}(j) \cup \text{ntr-}\mathcal{RS}\left(\frac{\sigma + 2n + 1}{1 - \gamma}\right).$$

(vii) If $\sigma \in (\gamma_m, \gamma_{m+1})$ for some $m \in \{0, 2, \dots, 2n-2\}$, then

$$\mathcal{RS} = \bigcup_{j=m+1}^{2n} \text{tr-}\mathcal{RS}(j).$$

(viii) If $\sigma = \gamma_m$ for some $m \in \{2, 4, \ldots, 2n\}$ and $Q_m < \infty$, then

$$\mathcal{RS} = \bigcup_{j=m}^{2n} \text{tr-}\mathcal{RS}(j) \cup \text{ntr-}\mathcal{RS}(m).$$

(ix) If $\sigma = \gamma_m$ for some $m \in \{2, 4, \ldots, 2n-2\}$ and $Q_m = \infty$, then

$$\mathcal{RS} = \bigcup_{j=m+1}^{2n} \text{tr-}\mathcal{RS}(j).$$

(x) If $\sigma = \gamma_{2n}$ and $Q_{2n} = \infty$, then

$$\mathcal{RS} = \emptyset.$$

(xi) If $\sigma > \gamma_{2n}$, then

$$\mathcal{RS} = \emptyset.$$

A similar description was obtained for the equation

$$y^{2n+1} = p(t)\Phi_\gamma(y),$$

where $0 < \gamma < 1$ and $p : [a, \infty) \to (0, \infty)$ is a continuous $\mathcal{RV}$ function, in the paper [90] by Kusano and Manojlović.

For further related information see Remark 4.5-(v).

4.3.7 $\mathcal{P}_\omega$ solutions of n-th order equations

In this section we present a selected result from Evtukhov, Samoilenko [36] which represents the approach of Evtukhov et al. to the investigation of Emden-Fowler equation in the framework close to regular variation.

Consider the n-th order differential equation

$$y^{(n)} = \delta p(t)F(y), \tag{4.59}$$

where $\delta \in \{-1, 1\}$, $p : [a, \omega) \to (0, \infty)$ is a continuous function, $-\infty < a < \omega \leq \infty$, and $F : \Delta_T \to (0, \infty)$ is a continuous $\mathcal{RV}_T$ function with the index $\alpha \neq 1$; here T is zero or $\pm\infty$, and Δ_T is an one-sided neighborhood of T. By $F \in \mathcal{RV}_T(\alpha)$ we mean here that

$$F(u) = |u|^\alpha L_F(u) \;\text{ with }\; \lim_{u \to T,\, u \in \Delta_T} \frac{L_F(\lambda u)}{L_F(u)}$$

for each $\lambda > 0$.

A solution y of (4.59) is called a $\mathcal{P}_\omega(T,\lambda_0)$ solution, $-\infty \le \lambda_0 \le \infty$, if it is defined on an interval $[t_0,\omega) \subset [a,\omega)$ and satisfies the conditions

$$y : [t_0,\omega) \to \Delta_T, \quad \lim_{t\to\omega} y(t) = T, \tag{4.60}$$

$$\lim_{t\to\omega} y^{(k)}(t) = \begin{cases} \text{either } 0 \\ \text{or } \pm\infty \end{cases} \quad (k = 1,\dots,n),$$

$$\lim_{t\to\omega} \frac{(y^{(n-1)}(t))^2}{y^{(n)}(t)y^{(n-2)}(t)} = \lambda_0. \tag{4.61}$$

Take a number $b \in \Delta_T$ such that $|b| < 1$ if $T = 0$, $b > 1$ if $T = \infty$, $b < -1$ if $T = -\infty$ and set $\Delta_T(b) = [b,T)$ if Δ_T is a left neighborhood of T, $\Delta_T(b) = (T,b]$ if Δ_T is a right neighborhood of T. It follows from the definition that each $\mathcal{P}_\omega(T,\lambda_0)$ solution of equation (4.59) and all of its derivatives of order less than or equal to n are nonzero on some interval $[t_1,\omega) \subset [t_0,\omega)$; moreover, the first derivative of the solution is positive on this interval if Δ_T is a left neighborhood of T and negative otherwise. With regard for this fact and the choice of b, we introduce two numbers

$$\mu_0 = \operatorname{sgn} b, \quad \mu_1 = \begin{cases} 1 & \text{if } \Delta_T \text{ is a left neighborhood of } T \\ -1 & \text{if } \Delta_T \text{ is a right neighborhood of } T, \end{cases}$$

which determine the signs of the $\mathcal{P}_\omega(T,\lambda_0)$ solution and its first derivative, respectively, on the interval $[t_1,\omega)$. In view of (4.60), we assume that $y(t) \in \Delta_T(b)$ for $t \in [t_1,\omega)$. Further, we put

$$t_\omega = \begin{cases} t & \text{for } \omega = \infty \\ t-\omega & \text{for } \omega < \infty. \end{cases}$$

Set

$$J(t) = \int_K^t s_\omega^{n-1} p(s)\,\mathrm{d}s, \text{ where } K = \begin{cases} a & \text{if } \int_a^\omega s_\omega^{n-1}p(s)\,\mathrm{d}s = \infty \\ \omega & \text{if } \int_a^\omega s_\omega^{n-1}p(s)\,\mathrm{d}s < \infty \end{cases}$$

Theorem 4.13. *Let*

$$\lambda_0 = \mathbb{R} \setminus \left\{0, \frac{1}{2}, \frac{2}{3}, \dots, \frac{n-2}{n-1}, 1\right\}.$$

Then for the existence of $\mathcal{P}_\omega(T,\lambda_0)$ *solutions of* (4.59), *it is necessary and, if the algebraic equation*

$$(1+\varrho)\prod_{i=1}^{n-1}(A_i+\varrho) = \alpha\prod_{i=1}^{n-1}A_i \tag{4.62}$$

for ϱ, *where* $A_i = (n-i)\lambda_0 - (n-i-1)$, $i = 1,\dots,n-1$, *has no roots with zero real part, sufficient that*

$$\lim_{t\to\omega}\frac{t_\omega J'(t)}{J(t)} = \frac{(1-\alpha)A_1}{\lambda_0 - 1} \tag{4.63}$$

and the following inequalities hold

$$\delta\mu_0[(\lambda_0-1)t_\omega]^n\prod_{i=1}^{n-1}A_i>0,\quad \mu_0\mu_1A_1(\lambda_0-1)t_\omega>0. \tag{4.64}$$

Moreover, there exists an m-parameter family of such solutions if, among the roots of the algebraic equation, there are m roots (with regard of multiplicities) whose real part has the same sign as the function $(1-\lambda_0)t_\omega$*. Furthemore, each solution of this kind admits the asymptotic representations*

$$\frac{y(t)}{F(y(t))}=\delta[(\lambda_0-1)t_\omega]^n p(t)\prod_{i=1}^{n-1}\frac{1}{A_i}(1+o(1))\quad \textit{as } t\to\omega, \tag{4.65}$$

$$\frac{y^{(k)}(t)}{y^{(k-1)}(t)}=\frac{A_k}{(\lambda_0-1)t_\omega}(1+o(1)),\ k=1,\ldots,n-1,\quad \textit{as } t\to\omega. \tag{4.66}$$

Proof. First we introduce the useful function

$$G(u)=\int_B^u\frac{\mathrm{d}s}{F(s)},\quad B=\begin{cases}b & \text{for } \left|\int_B^T\frac{\mathrm{d}s}{F(s)}\right|=\infty\\ T & \text{for } \left|\int_B^T\frac{\mathrm{d}s}{F(s)}\right|<\infty.\end{cases}$$

Note its properties needed in the sequel. Since $G'(u)>0$ for $u\in\Delta_T(b)$, we have $G:\Delta_T(b)\to\Delta_Z(c)$, where

$$\Delta_Z(c)=\begin{cases}[c,Z) & \text{for } \Delta_T(b)=[b,T)\\ (Z,c] & \text{for } \Delta_T(b)=(T,b],\end{cases}$$

$$c=\int_B^b\frac{\mathrm{d}s}{F(s)},\quad Z=\begin{cases}0 & \text{for } B=T\\ \infty & \text{for } B=b<T\\ -\infty & \text{for } B=b>T,\end{cases}$$

and moreover,

$$\lim_{u\to T,u\in\Delta_T(b)}F(u)=\frac{\mu_0}{1-\alpha}\lim_{u\to T,\,u\in\Delta_T(b)}|u|^{1-\alpha}=Z, \tag{4.67}$$

and there exists an inverse continuously differentiable increasing function $G^{-1}:\Delta_Z(c)\to\Delta_T(b)$ such that

$$\lim_{z\to Z,\,z\in\Delta_Z(c)}G^{-1}(z)=T. \tag{4.68}$$

By virtue of properties of $\mathcal{RV}_T$ functions and the L'Hospital rule, we have the relation

$$\lim_{u\to T,\,u\in\Delta_T(b)}\frac{u}{G(u)F(u)}=1-\alpha. \tag{4.69}$$

Necessity. Let

$$\lambda_0 = \mathbb{R} \setminus \left\{0, \frac{1}{2}, \frac{2}{3}, \dots, \frac{n-2}{n-1}, 1\right\}$$

and let $y : [t_0, \omega) \to \Delta_T$, be an arbitrary $\mathcal{P}_\omega(T, \lambda_0)$ solution of equation (4.59). Then there exists $t_1 \in [t_0, \omega)$ such that $y(t) \in \Delta_T(b)$, $\operatorname{sgn} y(t) = \mu_0$, and $\operatorname{sgn} y'(t) = \mu_1$ for $t \in [t_1, \omega)$. In addition, by [36, Lemma 2.1], we have the asymptotic relations

$$y^{(k-1)}(t) \sim [(\lambda_0 - 1)t_\omega]^{n-k} \prod_{i=1}^{n-1} \frac{1}{A_i} y^{(n-1)}(t),\ k = 1, \dots, n-1, \quad \text{as } t \to \omega,$$

where $A_i \neq 0$, $i = 1, \dots, n-1$. They imply asymptotic representations (4.66), and, by virtue of (4.61), we have

$$\begin{aligned} y^{(n)}(t) &\sim \frac{1}{\lambda_0}\left(\frac{y^{(n-1)}(t)}{y^{(n-2)}(t)}\right)^2 y^{(n-2)}(t) = \frac{1}{\lambda_0}\left(\frac{y^{(n-1)}(t)}{y^{(n-2)}(t)}\right)^2 \frac{y^{(n-2)}(t)}{y^{(n-3)}(t)} \cdots \frac{y''(t)}{y'(t)} y'(t) \\ &\sim \frac{\lambda_0}{(\lambda_0 - 1)^2 t_\omega^2} \cdot \frac{A_{n-2}}{(\lambda_0 - 1)t_\omega} \cdots \frac{a_2}{(\lambda_0 - 1)t_\omega} y'(t) = \frac{\prod_{i=2}^{n-1} A_i}{[(\lambda_0 - 1)t_\omega]^{n-1}} y'(t) \end{aligned}$$

as $t \to \omega$. Therefore by virtue of (4.59), we have (4.65). By integrating this asymptotic relation from t_1 to t, $t \in (t_1, \omega)$, and by taking into account conditions (4.60) and (4.67), we obtain the relation

$$G(y(t)) = \delta(\lambda_0 - 1)^{n-1} \prod_{i=2}^{n-1} \frac{1}{A_i} J(t)[1 + o(1)]$$

as $t \to \omega$, which, together with (4.69), implies the representation

$$\delta(\lambda_0 - 1)^{n-1} \prod_{i=2}^{n-1} \frac{1}{A_i} J(t)[1 + o(1)] \tag{4.70}$$

as $t \to \omega$. It follows from (4.65) and (4.70) that

$$\frac{y'(t)}{y(t)} = \frac{t_\omega^{n-1} p(t)}{(1 - \alpha)J(t)}(1 + o(1))$$

as $t \to \omega$. Therefore, by virtue of (4.66) for $k = 1$, we have

$$\frac{A_1}{(\lambda_0 - 1)t_\omega} = \frac{t_\omega^{n-1} p(t)}{(1 - \alpha)J(t)}(1 + o(1))$$

as $t \to \omega$. Consequently, condition (4.63) is satisfied, and relation (4.70) can be represented in the form (4.65). In addition, since $\operatorname{sgn} y(t) = \mu_0$ and $\operatorname{sgn} y'(t) = \mu_1$ for $t \in [t_1, \omega)$, it follows from (4.65) and (4.66) that inequalities (4.64) hold.

Sufficiency. Let conditions (4.63) and (4.64) be satisfied for some

$$\lambda_0 = \mathbb{R} \setminus \left\{0, \frac{1}{2}, \frac{2}{3}, \dots, \frac{n-2}{n-1}, 1\right\},$$

and let the algebraic equation (4.62) have no roots with zero real part. Let us show that, in this case, (4.59) has $\mathcal{P}_\omega(T, \lambda_0)$ solutions that admit asymptotic representations (4.65) and (4.66). By applying the transformation

$$F(y(t)) = q(t)(1 + v_1(\tau)), \quad \frac{y^{(k)}(t)}{y^{(k-1)}(t)} = \frac{A_k}{(\lambda_0 - 1)t_\omega}(1 + v_{k+1}(\tau)), \tag{4.71}$$

$k = 1, \dots, n-1$, to equation (4.59), where

$$\tau = \beta \ln|t_\omega|, \quad \beta = \begin{cases} 1 & \text{for } \omega = \infty \\ -1 & \text{for } \omega < \infty, \end{cases} \quad q(t) = \delta(\lambda_0 - 1)^{n-1} \prod_{i=2}^{n-1} \frac{1}{A_i} J(t),$$

we obtain the system of differential equations

$$\begin{aligned} v_1' &= \beta\left(\frac{t_\omega J'(t)}{J(t)}(1 + v_1) + \frac{A_1}{\lambda_0 - 1} \cdot \frac{Y(t, v_1)}{F(Y(t, v_1))q(t)}(1 + v_2)\right), \\ v_k' &= \beta\left(1 + v_k + \frac{A_k}{\lambda_0 - 1}(1 + v_k)(1 + v_{k+1}) - \frac{A_{k-1}}{\lambda_0 - 1}(1 + v_k)^2\right), \\ v_n' &= \beta\left(1 + v_n - \frac{A_{n-1}}{\lambda_0 - 1}(1 + v_n)^2 + \frac{p(t)t_\omega^n}{A_1 J(t)} \cdot \frac{q(t)F(Y(t, v_1))}{Y(t, v_1)\prod_{k=2}^{n-1}(1 + v_k)}\right), \end{aligned} \tag{4.72}$$

$k = 2, \dots, n-1$, where $t = t(\tau)$ is the inverse function of $\tau = \beta \ln|t_\omega|$ and

$$Y(t, v_1) = G^{-1}(q(t)(1 + v_1)).$$

By virtue of conditions (4.63) and (4.64), $\lim_{t\to\omega} q(t) = Z$, and there exists a number $t_0 \in [a, \omega)$ such that $q(t)(1 + v_1) \in \Delta_Z(c)$ for $t \in [t_0, \omega)$ and $|v_1| \le 1/2$. Therefore, $Y(t, v_1) \in \Delta_T(b)$ for $t \in [t_0, \omega)$ and, by virtue of (4.68),

$$\lim_{t\to\omega} Y(t, v_1) = T \;\text{ uniformly with respect to } v_1 \in [-1/2, 1/2].$$

This, together with (4.69), implies that

$$\lim_{t\to\omega} \frac{Y(t, v_1)}{F(Y(t, v_1))G(Y(t, v_1))} = 1 - \alpha$$

uniformly with respect to $v_1 \in [-1/2, 1/2]$, i.e.,

$$\frac{Y(t, v_1)}{F(Y(t, v_1))} = [1 - \alpha + R_1(t, v_1)]G(Y(t, v_1))$$

and

$$\frac{F(Y(t,v_1))}{Y(t,v_1)} = \frac{1/(1-\alpha)+R_2(t,v_1)}{F(Y(t,v_1))},$$

where the functions R_i, $i=1,2$, satisfy the conditions

$$\lim_{t\to\omega} R_i(t,v_1)=0,\ i=1,2, \tag{4.73}$$

uniformly with respect to $v_1 \in [-1/2,1/2]$. Therefore, by virtue of the form of the function $Y(t,v_1)$, we have the representations

$$\begin{aligned}\frac{Y(t,v_1)}{F(Y(t,v_1))} &= [1-\alpha+R_1(t,v_1)]q(t)(1+v_1),\\ \frac{F(Y(t,v_1))}{Y(t,v_1)} &= \frac{1/(1-\alpha)+R_2(t,v_1)}{q(t)(1+v_1)}.\end{aligned} \tag{4.74}$$

By taking into account these representations and by setting

$$h(t)=\frac{(\lambda_0-1)t_\omega J'(t)}{A_1(1-\alpha)J(t)},$$

we rewrite system (4.72) in the form

$$\begin{aligned} v_1' &= \frac{\beta}{\lambda_0-1}[f_1(\tau,v_1,v_2)+A_1(1-\alpha)v_2+V_1(v_1,v_2)],\\ v_k' &= \frac{\beta}{\lambda_0-1}[-A_{k-1}v_k+A_0v_{k+1}+V_k(v_k,v_{k+1})],\ \ k=2,\dots,n-1,\\ v_n' &= \frac{\beta}{\lambda_0-1}\left(f_2(\tau,v_1,\dots,v_{n-1})-\sum_{i=1}^{n-1}v_i-(\lambda_0+1)v_n+V_n(\tau,v_1,\dots,v_n)\right),\end{aligned} \tag{4.75}$$

where

$$\begin{aligned} &f_1(\tau,v_1,v_2)=A_1[R_1(t,v_1)(1+v_2)+(1-\alpha)-(1-\alpha)h(t)](1+v_1),\\ &V_1(v_1,v_2)=A_1(1-\sigma)v_1v_2,\\ &V_k(v_k,v_{k+1})=A_kv_kv_{k+1}-A_{k-1}v_k^2,\ \ k=2,\dots,n-1,\\ &f_2(\tau,v_1,\dots,v_{n-1})=(1-\alpha)h(t)R_2(t,v_1)\prod_{k=1}^{n-1}\frac{1}{1+v_k}+(h(t)-1)\left(1-\sum_{k=1}^{n-1}v_k\right),\\ &V_n(\tau,v_1,\dots,v_n)=-A_{n-1}v_n^2+h(t)\left(\prod_{k=1}^{n-1}\frac{1}{1+v_k}-1+\sum_{k=1}^{n-1}v_k\right)\end{aligned}$$

Consider the resulting system on the set $[\tau_0,\infty)\times\mathbb{R}^n_{1/2}$, where $\tau_0=\beta\ln|t_\omega|$ and $\mathbb{R}^n_{1/2}=\{(v_1,\dots,v_n)\in\mathbb{R}^n : |v_k|\le 1/2, k=1,\dots,n\}$. Since $\tau'(t)=\beta/t_\omega>0$ for $t\in[t_0,\omega)$, $\lim_{t\to\omega}\tau(t)=\infty$, and $t=t(\tau)$ is the inverse function of $\tau:[t_0,\omega)\to[\tau_0,\infty)$, it follows that the right-hand sides of the system are continuous on the set

$[\tau_0,\infty)\times\mathbb{R}^n_{1/2}$, and, by virtue of the conditions of the theorem, $\lim_{\tau\to\infty} f_1(\tau,v_1,v_2) = 0$ uniformly with respect to $(v_1,v_2)\in\mathbb{R}^2_{1/2}$, $\lim_{\tau\to\infty} f_2(\tau,v_1,\ldots,v_{n-1}) = 0$ uniformly with respect to $(v_1,\ldots,v_{n-1})\in\mathbb{R}^{n-1}_{1/2}$,

$$\lim_{|v_k|+|v_{k+1}|\to 0}\frac{V_k(v_k,v_{k+1})}{|v_k|+|v_{k+1}|}=0$$

$k=1,\ldots,n-1$,

$$\lim_{|v_1|+\cdots+|v_n|\to 0}\frac{V_n(\tau,v_1,\ldots,v_n)}{|v_1|+\cdots+|v_n|}=0$$

uniformly with respect to $\tau\in[\tau_0,\infty)$. In addition, by writing out the characteristic equation $\det(C-\varrho E)=0$ of the matrix C consisting of the coefficients $v_1,\ldots,v_n$ in square brackets in system (4.75), that is, the matrix

$$C=\begin{pmatrix} 0 & A_1(1-\alpha) & 0 & \cdots & 0 & 0\\ 0 & -A_1 & A_2 & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & -A_{n-2} & A_{n-1}\\ -1 & -1 & -1 & \cdots & -1 & -(\lambda_0+1)\end{pmatrix},$$

we obtain the algebraic equation (4.62), which, by the assumptions of the theorem, has no roots with zero real part. We have thereby shown that system (4.75) satisfies all assumptions of [35, Theorem 2.2]. By this theorem, system (4.75) has at least one solution $(v_1,\ldots,v_n):[\tau_1,\infty)\to\mathbb{R}^n$, $\tau_1\ge\tau_0$, that tends to zero as $\tau\to\infty$. Moreover, there exists an m-parameter family of such solutions if, among the roots of algebraic equation (4.62), there are m roots whose signs coincide with that of $\beta(1-\lambda_0)$. By virtue of the changes of variables (4.71) and conditions (4.69) and (4.63), to each such solution of system (4.75), there corresponds a solution of differential equation (4.59) admitting asymptotic representations (4.65) and (4.66). One can readily see that this solution is a $\mathcal{P}_\omega(T,\lambda_0)$ solution of (4.59). Since $\beta=\operatorname{sgn} t_\omega$, it follows that equation (4.59) has an m-parameter family of such solutions if, among the roots of (4.62), there are m roots whose sign coincide with that of the function $(1-\lambda_0)t_\omega$ in a left neighborhood of ω. □

Remark 4.5. (i) Algebraic equation (4.62) necessarily has no solutions with zero real part if $|\alpha|<1$.

(ii) Assume that F satisfies the condition

$$L_F(zg(z)) = L_F(z)[1+o(1)] \quad \text{as } z\to T\ (z\in\Delta_T) \tag{4.76}$$

for every continuously differentiable function $g:\Delta_T\to(0,\infty)$ with the property

$$\lim_{z\to T,\, z\in\Delta_T}\frac{zg'(z)}{g(z)}=0.$$

Compare (4.76) with (4.23), (4.95), and (4.96). If $y : [t_0, \omega) \to \Delta_T$ is a continuously differentiable function such that

$$\lim_{t\to\omega} y(t) = T, \quad \frac{y'(t)}{y(t)} = \frac{\xi'(t)}{\xi(t)}[\vartheta + o(1)] \text{ as } t \to \omega,$$

where ϑ is a nonzero real constant and ξ is a real function differentiable in some left neighborhood of ω and such that $\xi'(t) \neq 0$, then

$$L_F(y(t)) = L_F(\mu_0|\xi(t)|^{\vartheta})[1 + o(1)]$$

as $t \to \omega$, since $y(t) = z(t)g(z(t))$, where $z(t) = \mu_0|\xi(t)|^{\vartheta}$, and

$$\begin{aligned}\lim_{z\to T,\, z\in\Delta_T} \frac{zg'(z)}{g(z)} &= \lim_{t\to\omega} \frac{z(t)g'(z(t))}{g(z(t))} \\ &= \lim_{t\to\omega} \frac{z(t)(y(t)/z(t))'}{(y(t)/z(t))z'(t)} \\ &= \lim_{t\to\omega} \left(\frac{\xi(t)y'(t)}{\vartheta\xi'(t)y(t)} - 1\right) = 0.\end{aligned}$$

By virtue of representations (4.66), a $\mathcal{P}_\omega(T, \lambda_0)$ solution of (4.59) is a function regularly varying (at ω) of index $\vartheta = A_1/(\lambda_0 - 1)$. Therefore, if in Theorem 4.13 we additionally assume that F satisfies (4.76), then the above observations, together with (4.65), imply the asymptotic representation

$$y(t) = \mu_0 \left| [(\lambda_0 - 1)t_\omega]^n p(t) \prod_{i=1}^{n-1} \frac{1}{A_i} L_F\left(\mu_0 |t_\omega|^{\frac{A_1}{\lambda_0 - 1}}\right) \right|^{\frac{1}{1-\alpha}} (1 + o(1))$$

as $t \to \omega$. Therefore, in this case, the asymptotic formulas for $\mathcal{P}_\omega(T, \lambda_0)$ solutions and their derivatives of order less than or equal to $n-1$ can be written out in closed form.

(iii) Note that Evtukhov et al. in their previous papers related to the topic of this chapter (some of them are mentioned above), additionally assumed that F is a twice continuously differentiable function on the interval Δ_T and

$$\lim_{u\to T,\, u\in\Delta_T} \frac{uF''(u)}{F'(u)} = \alpha - 1.$$

However, in the result which we describe here, the approach was improved in such a way that this condition can be omitted.

(iv) The paper [36] also deals with other types of solutions; in particular conditions guaranteeing the existence in the classes $\mathcal{P}_\omega(T, \zeta)$, where $\zeta = \pm\infty$ or $\zeta = 1$ or $\zeta = (n - 2)/(n - 1)$ or $\zeta = (n - i - 1)/(n - i)$ with $i = 2, 3, \ldots, n - 1$ or $\zeta = 0$, are established.

(v) In the paper [89] by Kusano and Manojlović, comments and comparisons of the results with ones in [36] are presented; note that a summary of the results from [89] is given in the previous section and concerns equation (4.58) which is a special case of (4.59). The observations which we present next, are based on those comments and somehow adapted. Although not specifically emphasized in [36] — with some exceptions — Evtukhov and Samoilenko restricted their attention on the equation with $\mathcal{RV}$ coefficient and focused their attention only on its $\mathcal{RV}$ solutions. Namely, conditions imposed in [36] on the function p in (4.59) mean, due to converse half of Karamata's theorem, that p is of regular variation. That fact is neither mentioned nor used by Evtukhov and Samoilenko. Moreover, while they clearly emphasized that $\mathcal{P}_\infty(T,\lambda_0)$ solutions with

$$\lambda_0 = \mathbb{R} \setminus \left\{0, \frac{1}{2}, \frac{2}{3}, \dots, \frac{n-2}{n-1}, 1\right\},$$

and $\lambda_0 = \infty$ are functions of regular variation (cf. (ii) of this remark), such an assertion for $\mathcal{P}_\infty(T,(n-i-1)/(n-i))$ solutions with $i \in \{1,2,\dots,n-1\}$ is missing. Also, by virtue of representations of such solutions obtained in [36] such a conclusion is almost impossible to make. But, using the fact that p is of regular variation, it becomes quite clear that each $\mathcal{P}_\infty(T,\lambda_0)$ solution, with $\lambda_0 \neq 1$, is regularly varying, while assuming that p is of rapid variation, $\mathcal{P}_\infty(T,1)$ solutions are rapidly varying. Moreover, the converse is also true, due to the fact that $p \in \mathcal{RV}(\sigma)$. Indeed, denote with $\mathcal{P}(T,\lambda_0)$ the set of all $\mathcal{P}_\infty(T,\lambda_0)$ solutions of (4.58). Then, assuming that $y \in \mathcal{RV}(\vartheta)$, from equation (4.58) we may conclude $y^{(n)} \in \mathcal{RV}(\sigma+\vartheta\gamma)$, which by application of the Karamata integration theorem implies that

$$y^{(n-1)} \in \mathcal{RV}(\sigma+\vartheta\gamma+1) \;\Rightarrow\; \lim_{t\to\infty} t\frac{y^{(n)}(t)}{y^{(n-1)}(t)} = \sigma+\vartheta\gamma+1$$
$$y^{(n-2)} \in \mathcal{RV}(\sigma+\vartheta\gamma+2) \;\Rightarrow\; \lim_{t\to\infty} t\frac{y^{(n-1)}(t)}{y^{(n-2)}(t)} = \sigma+\vartheta\gamma+2.$$

Therefore, condition (4.61) in the definition of $\mathcal{P}_\infty(T,\lambda_0)$ solutions becomes

$$\frac{\sigma+\vartheta\gamma+2}{\sigma+\vartheta\gamma+1} = \lim_{t\to\infty} \frac{t\frac{y^{(n-1)}(t)}{y^{(n-2)}(t)}}{t\frac{y^{(n)}(t)}{y^{(n-1)}(t)}} = \lim_{t\to\infty} \frac{[y^{(n-1)}(t)]^2}{y^{(n)}y^{(n-2)}(t)},$$

so that

$$y \in \mathcal{P}\left(T, \frac{\sigma+\vartheta\gamma+2}{\sigma+\vartheta\gamma+1}\right), \text{ i.e., } \mathcal{RS}(\vartheta) \subseteq \mathcal{P}\left(T, \frac{\sigma+\vartheta\gamma+2}{\sigma+\vartheta\gamma+1}\right),$$

where $\mathcal{RS}(\vartheta) = \mathcal{RS} \cap \mathcal{RV}(\vartheta)$, $\mathcal{RS}$ being the set of all $\mathcal{RV}$ solutions of (4.58). More specifically, it is not difficult to see that, in fact

$$
\begin{aligned}
\mathcal{RS}(\vartheta) =& \mathcal{P}\left(T, \frac{2n-1-\vartheta}{2n-\vartheta}\right), \\
& \qquad \text{for } \vartheta \in (-\infty, 2n) \setminus \{0, 1, \dots, 2n-1\},\ T = \begin{cases} 0 & \text{if } \vartheta < 0 \\ \infty & \text{if } \vartheta > 0 \end{cases}, \\
\mathcal{RS}(0) =& \mathcal{P}\left(0, \frac{2n-1}{2n}\right), \\
\mathcal{RS}(k) =& \mathcal{P}\left(\infty, \frac{2n+1-k}{2n-k}\right) \ \text{ for } k = \{1, 2, \dots, 2n-1\}, \\
\mathcal{RS}(2n) =& \mathcal{P}(\infty, \infty),
\end{aligned}
$$

which makes the results from the previous subsection closely connected with the ones in this section and in [36].

4.3.8 Strongly monotone solutions of n-th order nonlinear systems

This subsection is based on the paper [124] by Matucci, Řehák and on the paper [146] by Řehák. Consider the nonlinear differential system

$$
\begin{cases}
x_1' & = \delta a_1(t) F_1(x_2), \\
x_2' & = \delta a_2(t) F_2(x_3), \\
 & \vdots \\
x_{n-1}' & = \delta a_{n-1}(t) F_{n-1}(x_n), \\
x_n' & = \delta a_n(t) F_n(x_1),
\end{cases}
\tag{4.77}
$$

$n \in \mathbb{N}$, $n \geq 2$, where F_i, $i = 1, \dots, n$, are continuous functions defined on $\mathbb{R}$ with $uF_i(u) > 0$ for $u \neq 0$, a_i, $i = 1, \dots, n$, are positive continuous functions defined on $[T, \infty)$, $T \geq 0$, and $\delta \in \{-1, 1\}$.

We assume

$$a_i \in \mathcal{RV}(\sigma_i), \ \ \sigma_i \in \mathbb{R}, \ \ i = 1, \dots, n, \tag{4.78}$$

and

$$F_i(|\cdot|) \in \mathcal{RV}(\alpha_i), \ \ \alpha_i \in (0, \infty), \ \ i = 1, \dots, n, \tag{4.79}$$

when studying the asymptotics of strongly increasing solutions, or (4.78) and

$$F_i(|\cdot|) \in \mathcal{RV}_0(\alpha_i), \ \ \alpha_i \in (0, \infty), \ \ i = 1, \dots, n. \tag{4.80}$$

when the asymptotics of strongly decreasing solutions is considered. Further, in both cases we assume that the indices $\alpha_1, \cdots, \alpha_n$ satisfy

$$\alpha_1 \cdots \alpha_n < 1. \tag{4.81}$$

Recall that system (4.77) satisfying condition (4.81) is called *subhomogeneous* (an alternative terminology is *sub-half-linear*). The opposite (strict) inequality is called *superhomogeneity* (or *super-half-linearity*).

The aim is to study asymptotic behavior of positive solutions to system (4.77), i.e., solutions having all their components eventually positive. With $\delta = 1$, positive solutions are eventually increasing, while with $\delta = -1$, positive solutions are eventually decreasing. If one of the components in either case tends to a nonzero real constant as the independent variable tends to infinity, then the asymptotic behavior is clear from a certain point of view; asymptotic formulas can be easily derived from the integral form of system (4.77). Therefore, we are interested in the extreme cases where all the solution components tend to infinity or tend to zero. Among other, we establish conditions guaranteeing that all solutions in these classes are regularly varying and satisfy certain asymptotic formula.

Let $\mathcal{S}$ denote the set of all solutions of (4.77) which are defined in a neighborhood of infinity and do not eventually vanish — the so-called *proper solutions* — and whose components are eventually positive. Due to the sign conditions on the coefficients and on the nonlinearities in (4.77), it is easy to see that if one component of a solution of (4.77) is eventually of one sign, then all its components are eventually of one sign; we speak about *nonoscillatory solutions*. Further, any nonoscillatory solution of (4.77) has necessarily all components eventually monotone. Denote by $\mathcal{DS} \subseteq \mathcal{S}$ the subset of the solutions of (4.77) which are in $\mathcal{S}$, and whose components are eventually decreasing; similarly, we denote by $\mathcal{IS} \subseteq \mathcal{S}$ the subset of the solutions of (4.77) which are in $\mathcal{S}$, and whose components are eventually increasing. We will study the set $\mathcal{DS}$ under the condition $\delta = -1$, and the set $\mathcal{IS}$ under the condition $\delta = 1$; later we explain why this setting is natural and nonrestrictive. First note that if $\delta = -1$, then $\mathcal{DS} = \mathcal{S}$, and if $\delta = 1$, then $\mathcal{IS} = \mathcal{S}$. It is clear that $\mathcal{DS}$ contains only those solutions whose components all tend to zero or at least one component tends to a positive constant and the other ones to zero (as $t \to \infty$), while $\mathcal{IS}$ contains only those solutions whose components tend all to infinity or at least one component tends to a positive constant and the other ones to infinity (as $t \to \infty$). Hence we denote

$$\mathcal{SDS} = \Big\{(x_1, \dots, x_n) \in \mathcal{DS} : \lim_{t\to\infty} x_i(t) = 0,\ i = 1, \dots, n\Big\},$$

which are the so-called *strongly decreasing solutions*, and

$$\mathcal{SIS} = \Big\{(x_1, \dots, x_n) \in \mathcal{IS} : \lim_{t\to\infty} x_i(t) = \infty,\ i = 1, \dots, n\Big\},$$

which are the so-called *strongly increasing solutions*.

The condition $\delta = 1$ resp. $\delta = -1$ is somehow natural when studying the classes $\mathcal{IS}$ resp. $\mathcal{DS}$. In order to justify this assertion, we recall a standard classification of nonoscillatory solutions. At the same time, thereby we put the presented results into a broader context. Naito in [128, 129] considers the n-th order differential equation

$$D(\gamma_n)D(\gamma_{n-1})\cdots D(\gamma_1)x + \tilde{\delta}p(t)\Phi_\beta(x) = 0, \tag{4.82}$$

where $n \geq 2$, $\gamma_1, \ldots, \gamma_n, \beta \in (0, \infty)$, $\tilde{\delta} = 1$ or $\tilde{\delta} = -1$, $p(t) > 0$, and $D(\gamma)x = \frac{d}{dt}(\Phi_\gamma(x))$. Equation (4.82) is a special case of system (4.77). Indeed, if $F_i = \Phi_{\alpha_i}$, $i = 1, \ldots, n$, then (4.77) can be equivalently written as

$$D_{a_{n-1}^{-\frac{1}{\alpha_{n-1}}}}\left(\frac{1}{\alpha_{n-1}}\right) \cdots D_{a_1^{-\frac{1}{\alpha_1}}}\left(\frac{1}{\alpha_1}\right) D_1(1)\, x_1 = \delta^n a_n(t) \Phi_{\alpha_n}(x_1), \tag{4.83}$$

where $D_f(\gamma)x = \frac{d}{dt}(f(t)\Phi_\gamma(x))$. Using the substitution $x_1(t) = \Phi_{\gamma_1}(x(t))$ and noticing that $D_1(1)(\Phi_{\gamma_1}(x)) = D(\gamma_1)x$, equation (4.83) reduces to (4.82) choosing $\alpha_i = 1/\gamma_{i+1}$, $i = 1, \ldots, n-1$, $\alpha_n = \beta/\gamma_1$, $a_i(t) \equiv 1$, $i = 1, \ldots, n-1$, $a_n(t) = p(t)$, and $\tilde{\delta} = -\delta^n$. Naito made a basic classification of positive solutions to (4.82) extending well known results by Kiguradze and Chanturia [79] for the quasilinear equation

$$x^{(n)} + \tilde{\delta} p(t) \Phi_\beta(x) = 0. \tag{4.84}$$

Notice that (4.82) reduces to (4.84) when $\gamma_1 = \cdots = \gamma_n = 1$. The classification of solutions x to (4.82) in Naito [128] is made according to the eventual signs of $D(\gamma_j) \cdots D(\gamma_1)x(t)$, $j = 0, \ldots, n-1$ (with the operator being the identity when $j = 0$). For solutions x of (4.82), the so-called *Kiguradze class of degree* k is denoted by $\mathcal{K}_k$ and defined as

$$\begin{cases} D(\gamma_j) \cdots D(\gamma_1)x(t) > 0 & \text{for } j = 0, 1, \ldots, k-1, \\ (-1)^{j-k} D(\gamma_j) \cdots D(\gamma_1)x(t) > 0 & \text{for } j = k, \ldots, n-1 \end{cases} \tag{4.85}$$

for t large, where for $k = 0$ (resp. $k = n$), the first line (resp. the second line) in (4.85) is omitted. It is not difficult to see that

$$x \in \mathcal{K}_0 \iff (x_1, \ldots, x_n) \in \mathcal{DS} \text{ and } x \in \mathcal{K}_n \iff (x_1, \ldots, x_n) \in \mathcal{IS}, \tag{4.86}$$

where $x_1 = \Phi_{\gamma_1}(x)$, $x_i = \delta \Phi_{\gamma_i}(x'_{i-1})$, $i = 2, \ldots, n$. If we denote the set of all eventually positive solutions of (4.82) by $\mathcal{K}$, then [128, Theorem 1.1] implies that

$$\begin{cases} \mathcal{K} = \mathcal{K}_1 \cup \mathcal{K}_3 \cup \cdots \cup \mathcal{K}_{n-1} & \text{for } \tilde{\delta} = 1 \text{ and } n \text{ even}; \\ \mathcal{K} = \mathcal{K}_0 \cup \mathcal{K}_2 \cup \cdots \cup \mathcal{K}_{n-1} & \text{for } \tilde{\delta} = 1 \text{ and } n \text{ odd}; \\ \mathcal{K} = \mathcal{K}_0 \cup \mathcal{K}_2 \cup \cdots \cup \mathcal{K}_{n-2} \cup \mathcal{K}_n & \text{for } \tilde{\delta} = -1 \text{ and } n \text{ even}; \\ \mathcal{K} = \mathcal{K}_1 \cup \mathcal{K}_3 \cup \cdots \cup \mathcal{K}_{n-2} \cup \mathcal{K}_n & \text{for } \tilde{\delta} = -1 \text{ and } n \text{ odd}. \end{cases} \tag{4.87}$$

In view of (4.86) and the equality $\delta^n a_n(t) = -\tilde{\delta} p(t)$, the relations in (4.87) says that the condition $\delta = -1$ and $\delta = 1$ are quite natural and nonrestrictive when studying, respectively, positive decreasing and positive increasing solutions of (4.77).

The subscripts that indicate the components are always to be intended modulo n and not bigger than n, that is

$$u_k = \begin{cases} u_k & \text{if } 1 \leq k \leq n, \\ u_{k-mn} & \text{if } k > n, \end{cases} \tag{4.88}$$

where $m \in \mathbb{N}$ is such that $1 \le k - mn \le n$. With this convention, which is used throughout the paper, system (4.77) takes the form

$$x_i' = \delta a_i(t) F_i(x_{i+1}), \quad i = 1, \dots, n.$$

As usual, the slowly varying component in the representation of $f \in \mathcal{RV}(\vartheta)$ will be denoted by L_f, i.e., $L_f(t) = f(t)/t^{\vartheta}$; similarly for $f \in \mathcal{RV}_0(\vartheta)$.

For sake of simplicity, we introduce here some constants that repeatedly appear in what follows. We set

$$A_{i,j} = \prod_{k=i}^{j-1} \alpha_k, \text{ for } 1 \le i \le j \le i + n \le 2n.$$

It is easy to check that $A_{i,n+i} = \alpha_1 \cdots \alpha_n$ and $A_{i,i} = 1$ for all $i = 1, \dots, n$. We emphasize that the convention (4.88) is used only for simple subscripts and not for double ones. We indicate by $(\nu_1, \dots, \nu_n)$ the unique solution of the linear system

$$\nu_i - \alpha_i \nu_{i+1} = \sigma_i + 1, \quad i = 1, \dots, n, \tag{4.89}$$

where $\sigma_1, \dots, \sigma_n$ are given real numbers. Notice that the associated matrix is nonsingular thanks to the subhomogeneity condition. Further, let $(h_1, \dots, h_n)$ be the unique solution of

$$|\nu_i| h_i = h_{i+1}^{\alpha_i}, \quad i = 1, \dots, n. \tag{4.90}$$

Notice that the subhomogeneity condition plays a key role in its unique solvability again. A simple calculation shows that

$$\nu_i = \frac{1}{1 - A_{i,n+i}} \sum_{k=0}^{n-1} (\sigma_{i+k} + 1) A_{i,i+k}, \quad i = 1, \dots, n, \tag{4.91}$$

and

$$h_i = \left(\prod_{k=0}^{n-1} |\nu_{i+k}|^{-A_{i,i+k}} \right)^{\frac{1}{1-A_{i,n+i}}}, \quad i = 1, \dots, n.$$

If we set

$$L_i(t) = \left(\prod_{j=0}^{n-1} \left(L_{a_{i+j}}(t) L_{F_{i+j}}(t^{\nu_{i+j+1}}) \right)^{A_{i,i+j}} \right)^{\frac{1}{1-A_{i,n+i}}}, \quad i = 1, \dots, n, \tag{4.92}$$

then $(L_1, \dots, L_n)(t)$ is the unique solution (up to asymptotic equivalence) to the system of the relations

$$L_i(t) \sim L_{a_i}(t) L_{i+1}^{\alpha_i}(t) L_{F_i}(t^{\nu_{i+1}}) \quad \text{as } t \to \infty, \quad i = 1, \dots, n. \tag{4.93}$$

Observe that if $L_{F_1} \equiv \cdots \equiv L_{F_n} \equiv 1$, then $(L_1, \dots, L_n)(t)$ reduces to the unique solution of the system

$$L_i(t) = L_{a_i}(t) L_{i+1}^{\alpha_i}(t), \tag{4.94}$$

$i = 1, \dots, n$.

In what follows, we assume an additional condition for the slowly varying components of the nonlinearities F_i. In particular, if $F_i, i = 1, \dots, n$, satisfies (4.80), we assume

$$L_{F_i}(ug(u)) \sim L_{F_i}(u) \quad \text{as } u \to 0+, \quad i = 1, \dots, n, \tag{4.95}$$

for every $g \in \mathcal{SV}_0$, while if $F_i, i = 1, \dots, n$, satisfies (4.79), we assume

$$L_{F_i}(ug(u)) \sim L_{F_i}(u) \quad \text{as } u \to \infty, \quad i = 1, \dots, n, \tag{4.96}$$

for every $g \in \mathcal{SV}$. Compare conditions (4.95) and (4.96) with (4.23) and (4.76).

Now we are ready to present the main results. The first theorem gives sufficient conditions under which system (4.77) possesses a $\mathcal{SDS}$ or $\mathcal{SIS}$ solution which is regularly varying and we provide an exact asymptotic formula. We point out that F_i does not need to be monotone.

Theorem 4.14. *Let* (4.78) *hold.*

(i) Assume $\delta = -1$, (4.80) *and* (4.95). *If $\nu_i < 0$, $i = 1, \dots, n$, then there exists*

$$(x_1, \dots, x_n) \in \mathcal{SDS} \cap (\mathcal{RV}(\nu_1) \times \cdots \times \mathcal{RV}(\nu_n))$$

and

$$x_i(t) \sim h_i t^{\nu_i} L_i(t) \quad \textit{as } t \to \infty, \quad i{=}1,\dots,n. \tag{4.97}$$

(ii) Assume $\delta = 1$, (4.79) *and* (4.96) *If $\nu_i > 0$, $i = 1, \dots, n$, then there exists*

$$(x_1, \dots, x_n) \in \mathcal{SIS} \cap (\mathcal{RV}(\nu_1) \times \cdots \times \mathcal{RV}(\nu_n))$$

and (4.97) *holds.*

The proof of the above theorem will be given later. As a corollary, we get sufficient conditions for $\mathcal{SDS} \neq \emptyset$ and $\mathcal{SIS} \neq \emptyset$.

In the second result, strengthening the assumptions on the nonlinearities F_i, we can show that

all $\mathcal{SDS}$ and $\mathcal{SIS}$ solutions are regularly varying.

Theorem 4.15. *Let* (4.78) *hold and $F_i = \Phi_{\alpha_i}$ with $\alpha_i > 0$, $i = 1, \dots, n$.*

(i) If $\delta = -1$ and $\nu_i < 0$, $i = 1, \dots, n$, then $\mathcal{SDS} \neq \emptyset$ and for every $(x_1, \dots, x_n) \in \mathcal{SDS}$, it holds $(x_1, \dots, x_n) \in \mathcal{RV}(\nu_1) \times \cdots \times \mathcal{RV}(\nu_n)$. Further, (4.97) *holds with $L_{F_1} \equiv \cdots \equiv L_{F_n} \equiv 1$.*

(ii) If $\delta = 1$ and $\nu_i > 0$, $i = 1, \dots, n$, then $\mathcal{SIS} \neq \emptyset$ and for every $(x_1, \dots, x_n) \in \mathcal{SIS}$, it holds $(x_1, \dots, x_n) \in \mathcal{RV}(\nu_1) \times \cdots \times \mathcal{RV}(\nu_n)$. Further, (4.97) *holds with $L_{F_1} \equiv \cdots \equiv L_{F_n} \equiv 1$.*

An alternative expression of sufficient conditions in Theorem 4.14 and Theorem 4.15 can be found in subsequent Lemma 4.8.

In order to prove Theorem 4.14 and Theorem 4.15, we need some technical lemmas. The first two lemmas analyze conditions (4.95), (4.96) and show how they lead to the unique solvability (up to asymptotic equivalence) in the class $\mathcal{SV}$ of relation (4.93).

Lemma 4.3. *If* (4.95) *holds, then*

$$L_{F_i}(t^\nu h(t)) \sim L_{F_i}(t^\nu) \tag{4.98}$$

as $t \to \infty$ for any $\nu < 0$ and $h \in \mathcal{SV}$. Analogously, condition (4.96) *implies that* (4.98) *holds as $t \to \infty$ for any $\nu > 0$ and $h \in \mathcal{SV}$.*

Proof. Taking the substitution $u = t^\nu$, relation (4.98) is transformed into

$$L_{F_i}(uh(u^{-\nu})) \sim L_{F_i}(u), \quad u \to 0+. \tag{4.99}$$

Since $g(u) := h(u^{-\nu}) \in \mathcal{RV}_0(0) = \mathcal{SV}_0$, condition (4.95) implies that (4.99) holds. Similarly we can show that (4.96) implies (4.98) for any $\nu > 0$.

□

Lemma 4.4. *Let $\nu_i < 0$ for $i = 1, \dots, n$. Then the system of asymptotic relations* (4.93) *has a unique solution (up to asymptotic equivalence) belonging to the set $\mathcal{SV}_0$, and this solution is given by* (4.92)*. Further, the system of asymptotic relations*

$$\tilde{L}_i(t) \sim \frac{1}{|\nu_i|} L_{a_i}(t) \tilde{L}_{i+1}^{\alpha_i}(t) L_{F_i}(t^{\nu_{i+1}}) \quad \text{as } t \to \infty, \quad i = 1, \dots, n, \tag{4.100}$$

has the unique solution (up to asymptotic equivalence)

$$\tilde{L}_i(t) = h_i L_i(t),$$

where h_i satisfies (4.90)*, for all $i = 1, \dots, n$. Analogous statements hold under the assumption $\nu_i > 0$, $i = 1, \dots, n$.*

Proof. The first statement follows from the fact that system (4.93) can be written as

$$\begin{aligned} L_i(t) &\sim L_{a_i}(t) L_{F_i}(t^{\nu_{i+1}}) \left(L_{a_{i+1}}(t) L_{F_{i+1}}(t^{\nu_{i+2}}) L_{i+2}^{\alpha_{i+1}}(t) \right)^{\alpha_i} \sim \cdots \sim \\ &\sim \prod_{j=0}^{n-1} \left(L_{a_{i+j}}(t) L_{F_{i+j}}(t^{\nu_{i+j+1}}) \right)^{A_{i,i+j}} L_{i+n}^{A_{i,i+n}}(t) \end{aligned}$$

as $t \to \infty$, $i = 1, \dots, n$, and taking into account that $L_{i+n}(t) = L_i(t)$. The second statement is immediate from the definition of h_i, $i = 1, \dots, n$. Indeed it results

$$\tilde{L}_i(t) \sim \frac{h_{i+1}^{\alpha_i}}{|\nu_i|} L_{a_i}(t) L_{i+1}^{\alpha_i}(t) L_{F_i}(t^{\nu_{i+1}}) = \frac{1}{|\nu_i|} L_{a_i}(t) \tilde{L}_{i+1}^{\alpha_i}(t) L_{F_i}(t^{\nu_{i+1}})$$

.

□

The following lemma provides some properties of the constants ν_i, $i = 1, \dots, n$ defined by (4.89).

Lemma 4.5. *Let* $(p_1, \dots, p_n)$ *be the unique solution of the system*

$$\begin{cases} p_1 + \cdots + p_n = 1, \\ \alpha_i p_i + p_{i+2} = p_{i+1}(\alpha_{i+1} + 1), \ \ i = 1, 2, \dots, n-2, n. \end{cases} \tag{4.101}$$

Then

$$p_i = \frac{1 + \sum_{k=i+1}^{n+i-1} A_{k,n+i}}{n + \sum_{k=1}^{n}(A_{k,k+1} + A_{k,k+2} + \cdots + A_{k,k+n-1})} > 0, \quad i = 1, \dots, n, \tag{4.102}$$

and the following identity holds

$$\nu_1 + \cdots + \nu_n = \frac{(\sigma_1 p_1 + \cdots + \sigma_n p_n + 1)}{1 - \xi}, \tag{4.103}$$

where ξ *is defined by* $p_1 + \cdots + p_{n-2} + p_{n-1}(\alpha_{n-1} + 1)$ *and satisfies* $\xi < 1$. *Further, if* $L_{F_1} \equiv \cdots \equiv L_{F_n} \equiv 1$, *then*

$$L_1(t) \cdots L_n(t) = \left(L_{a_1}^{p_1}(t) \cdots L_{a_n}^{p_n}(t)\right)^{\frac{1}{1-\xi}} \tag{4.104}$$

Proof. It is easy to check that $(p_1, \dots, p_n)$ given by (4.102) solves (4.101). Positivity of p_i is clear and the uniqueness can readily be seen when writing (4.101) in a matrix form. The inequality $\xi < 1$ is equivalent to the subhomogeneity assumption. A series of routine and tedious computations (where we can expand the explicit expressions for ν_i and p_i, and compare corresponding summands in the resulting formulas) shows that identities (4.103) and (4.104) hold. □

Define

$$B_{i,j} = \begin{cases} A_{i,j} & \text{for } 1 \le i < j \le n, \\ \prod_{k=i}^{j-1+n} \alpha_k & \text{for } 1 \le j \le i \le j + n - 1 \le 2n - 1. \end{cases}$$

It is easy to verify that the following relations hold

$$\begin{aligned} B_{i,i} &= \alpha_1 \cdots \alpha_n, \\ B_{i,j} B_{j,i} &= \alpha_1 \cdots \alpha_n = B_{i,i}, \\ B_{i,j} B_{j,\ell} &= B_{i,\ell}, \end{aligned}$$

where $i, j, \ell \in \{1, \dots, n\}$, and for the last equality we assume $i < j < \ell$ or $\ell < i < j$ or $j < \ell < i$. Now let

$$\varrho_{i,j} = \nu_i - B_{i,j}\nu_j, \quad i, j \in 1, \dots, n. \tag{4.105}$$

In the subsequent lemmas we derive several properties of the constants $\varrho_{i,j}$, $i, j \in 1, \dots, n$. Their proofs are rather technical and since they do not directly concern regular variation we omit them. For details see [124].

Lemma 4.6. *Let* $\nu_i < 0$ *for* $i = 1, \dots, n$. *Then* $\varrho_{i,j}$, $i, j \in 1, \dots, n$, *satisfy the following relations:*

(i) $\varrho_{i,i} < 0$ *for* $i = 1, \dots, n$.
(ii) *If there exist* $i, j \in \{1, \dots, n\}$, $i \neq j$, *such that* $\varrho_{i,j} \geq 0$, *then* $\varrho_{j,i} < 0$.
(iii) *If there exist* $i, j, \ell \in \{1, \dots, n\}$, *with* $i < j < \ell$, *such that* $\varrho_{i,j} < 0$ *and* $\varrho_{j,\ell} \leq 0$, *then* $\varrho_{i,\ell} < 0$.
(iv) *If there exist* $i, j, \ell \in \{1, \dots, n\}$, *with* $j < i < \ell$, *such that* $\varrho_{i,j} < 0$ *and* $\varrho_{\ell,j} \geq 0$, *then* $\varrho_{i,\ell} < 0$.

Lemma 4.7. *Let* $\gamma_{i,j}$, $i, j = 1, \dots, n$, $n \geq 2$, *be any numbers which obey the rules (i)-(iv) in Lemma 4.6. Then the matrix* $(\gamma_{i,j})_{1 \leq i,j \leq n}$ *has at least one row whose elements are negative.*

The previous two lemmas play an important role in the proof of the following statement which, among other, enables us to find an alternative expression for sufficient conditions in Theorem 4.14 and Theorem 4.15.

Lemma 4.8. *The following equivalence holds:* $\nu_i < 0$ (> 0) *for all* $i = 1, \dots, n$ *if and only if* $\varrho_{i,m} < 0$ (> 0) *for all* $i = 1, \dots, n$ *and some* $m \in \{1, \dots, n\}$.

Lemma 4.9. *The numbers* $\varrho_{i,j}$, $i, j = 1, \dots, n$, *defined by* (4.105), *satisfy the relations*

$$\begin{cases} \varrho_{j+k-1,j} = \sigma_{j+k-1} + 1 + \alpha_{j+k-1}\varrho_{j+k,j}, & k = 1, \dots, n-1, \\ \varrho_{j+n-1,j} = \sigma_{j+n-1} + 1, \end{cases} \tag{4.106}$$

$j = 1, \dots, n$.

Now we are ready to prove the main theorems.

Proof of Theorem 4.14. (i) First we prove that (4.77) has at least one solution $\mathbf{x} = (x_1, \dots, x_n) \in \mathcal{SDS}$. Since (4.80) is valid, for every $i = 1, \dots, n$ there exists $\tilde{F}_i \in C(\mathbb{R}) \cap \mathcal{RV}_0(\alpha_i)$, nondecreasing, such that $\tilde{F}_i(u) \sim F_i(u)$ as $u \to 0$. Thus there exists $u_0 > 0$ such that

$$\frac{1}{\sqrt[4]{2}}\,\tilde{F}_i(u) \leq F_i(u) \leq \sqrt[4]{2}\,\tilde{F}_i(u), \quad \forall u \in [0, u_0], i = 1, \dots, n. \tag{4.107}$$

Notice that $\tilde{F}_i$ satisfies conditions (4.92) and (4.94) if F_i does, for $i = 1, \dots, n$. Let $(k_1, \dots, k_n)$, be the unique solution of the linear system $k_i - \alpha_i k_{i+1} = 1$. System (4.89) reduces to this if $\sigma_i = 0$ for all $i = 1, \cdots, n$, and therefore, from (4.91), it results $k_i > 0$ for all i. Let $\tilde{L}_i(t) = h_i L_i(t)$, $i = 1, \dots, n$, see Lemma 4.4. Now, taking into account that $\nu_i < 0$, $i = 1, \dots, n$, properties of $\mathcal{RV}$ functions, assumptions (4.98) and (4.100) imply that t_0 sufficiently large exists such that

$$2^{k_i}\tilde{L}_i(t)t^{\nu_i} \leq u_0, \tag{4.108}$$

$$L_{\tilde{F}_i}(2^{k_{i+1}}\tilde{L}_{i+1}(t)t^{\nu_{i+1}}) \leq \sqrt[4]{2}\,L_{\tilde{F}_i}(t^{\nu_{i+1}}), \tag{4.109}$$

$$L_{\tilde{F}_i}(2^{-k_{i+1}}\tilde{L}_{i+1}(t)t^{\nu_{i+1}}) \geq \frac{1}{\sqrt[4]{2}}\,L_{\tilde{F}_i}(t^{\nu_{i+1}}) \tag{4.110}$$

$$\frac{1}{\sqrt[4]{2}}\,\tilde{L}_i(t) \leq \frac{1}{|\nu_i|}\,L_{a_i}(t)\tilde{L}_{i+1}^{\alpha_i}(t)L_{\tilde{F}_i}(t^{\nu_{i+1}}) \leq \sqrt[4]{2}\,\tilde{L}_i(t), \tag{4.111}$$

$$\frac{1}{\sqrt[4]{2}}\,t^{\nu_i}\tilde{L}_i(t) \leq |\nu_i| \int_t^\infty s^{\nu_i - 1}\tilde{L}_i(s)\,\mathrm{d}s \leq \sqrt[4]{2}\,t^{\nu_i}\tilde{L}_i(t), \tag{4.112}$$

for all $t \in [t_0, \infty)$, and $i = 1, \dots, n$. Let $\Omega \subset (C[t_0, \infty))^n$ be the set

$$\Omega = \{(x_1, \dots, x_n) : \ x_i \in C[t_0, \infty), \\ 2^{-k_i}\tilde{L}_i(t)t^{\nu_i} \le x_i(t) \le 2^{k_i}\tilde{L}_i(t)t^{\nu_i}, \ i = 1, \dots, n\}, \quad (4.113)$$

and let $\mathbf{T} : \Omega \to (C[t_0, \infty))^n$ be the operator defined by

$$\mathbf{Tx} = (T_1x_2, T_2x_3, \dots, T_nx_1),$$

with

$$(T_ix_{i+1})(t) = \int_t^\infty a_i(s)F_i(x_{i+1}(s))\,\mathrm{d}s, \quad i = 1, \dots, n.$$

First of all notice that $\mathbf{T}$ is well defined in Ω. Indeed, for all $\mathbf{x} \in \Omega$ we have

$$0 \le a_i(t)F_i(x_{i+1}(t)) \le \sqrt[4]{2}\,a_i(s)\tilde{F}_i(x_{i+1}(t)) \le \sqrt[4]{2}\,a_i(s)\tilde{F}_i(2^{k_{i+1}}\tilde{L}_{i+1}(t)t^{\nu_{i+1}}),$$

where we used (4.108) and (4.107). Since $\tilde{F}_i \in \mathcal{RV}_0(\alpha_i)$ and $a_i \in \mathcal{RV}(\sigma_i)$, the last term in the above inequality belongs to the class $\mathcal{RV}(\sigma_i + \alpha_i\nu_{i+1}) = \mathcal{RV}(\nu_i - 1)$, see (4.89), with $\nu_i < 0$, and therefore it is integrable on $[t_0, \infty)$. In particular, taking into account (4.109), (4.111), and (4.112), for every $t \ge t_0$ it holds

$$\begin{aligned}
(T_ix_{i+1})(t) &\le \sqrt[4]{2}\int_t^\infty a_i(s)\tilde{F}_i(2^{k_{i+1}}\tilde{L}_{i+1}(s)s^{\nu_{i+1}})\,\mathrm{d}s \\
&= 2^{\alpha_ik_{i+1}+\frac{1}{4}}\int_t^\infty s^{\sigma_i}L_{a_i}(s)\tilde{L}_{i+1}^{\alpha_i}(s)s^{\alpha_i\nu_{i+1}}L_{\tilde{F}_i}(2^{k_{i+1}}\tilde{L}_{i+1}(s)s^{\nu_{i+1}})\,\mathrm{d}s \\
&= 2^{k_i-\frac{3}{4}}\int_t^\infty s^{\nu_i-1}L_{a_i}(s)\tilde{L}_{i+1}^{\alpha_i}(s)L_{\tilde{F}_i}(2^{k_{i+1}}\tilde{L}_{i+1}(s)s^{\nu_{i+1}})\,\mathrm{d}s \\
&\le 2^{k_i-\frac{1}{2}}\int_t^\infty s^{\nu_i-1}L_{a_i}(s)\tilde{L}_{i+1}^{\alpha_i}(s)L_{\tilde{F}_i}(s^{\nu_{i+1}})\,\mathrm{d}s \\
&\le 2^{k_i-\frac{1}{4}}\,|\nu_i|\int_t^\infty s^{\nu_i-1}\tilde{L}_i(s)\,\mathrm{d}s \le 2^{k_i}t^{\nu_i}\tilde{L}_i(t).
\end{aligned}$$

Analogously, from (4.110), (4.111), and (4.112), for every $t \ge t_0$ it holds

$$\begin{aligned}
(T_ix_{i+1})(t) &\ge 2^{-\frac{1}{4}}\int_t^\infty a_i(s)\tilde{F}_i(2^{-k_{i+1}}\tilde{L}_{i+1}(s)s^{\nu_{i+1}})\,\mathrm{d}s \\
&= 2^{-k_i+\frac{3}{4}}\int_t^\infty s^{\nu_i-1}L_{a_i}(s)\tilde{L}_{i+1}^{\alpha_i}(s)L_{\tilde{F}_i}(2^{-k_{i+1}}\tilde{L}_{i+1}(s)s^{\nu_{i+1}})\,\mathrm{d}s \\
&\ge 2^{-k_i+\frac{1}{2}}\int_t^\infty s^{\nu_i-1}L_{a_i}(s)\tilde{L}_{i+1}^{\alpha_i}(s)L_{\tilde{F}_i}(s^{\nu_{i+1}})\,\mathrm{d}s \\
&\ge 2^{-k_i+\frac{1}{4}}\,|\nu_i|\int_t^\infty s^{\nu_i-1}\tilde{L}_i(s)\,\mathrm{d}s \ge 2^{-k_i}t^{\nu_i}\tilde{L}_i(t).
\end{aligned}$$

Thus $\mathbf{T}$ maps Ω into itself. In order to apply the Schauder-Tychonoff fixed point theorem, we have to show that $\mathbf{T}$ is completely continuous. Since $\mathbf{T}(\Omega) \subseteq \Omega$, functions in $\mathbf{T}(\Omega)$ are equibounded on $[t_0, \infty)$; further, the inequality

$$0 \geq (T_i x_{i+1})'(t) \geq -\sqrt[4]{2}\, a_i(t)\tilde{F}_i(2^{k_{i+1}}\tilde{L}_{i+1}(t)t^{\nu_{i+1}})$$

which is valid for $t \geq t_0$, $i = 1, \dots, n$, and for every $(x_1, \dots, x_n) \in \Omega$, ensure that functions in $\mathbf{T}(\Omega)$ are equicontinuous. The relative compactness of $\mathbf{T}(\Omega)$ follows from the Ascoli-Arzelà theorem. To prove the continuity of $\mathbf{T}$ we have to show that for any sequence $\mathbf{x}^m = (x_1^m, \dots, x_n^m)$ in Ω which converges to $\bar{\mathbf{x}} = (\bar{x}_1, \dots, \bar{x}_n) \in \Omega$ as $m \to \infty$ uniformly on any compact subset of $[t_0, \infty)$, it holds $(\mathbf{T}\mathbf{x}^m)(t) \to (\mathbf{T}\bar{\mathbf{x}})(t)$ as $m \to \infty$ uniformly on compact subset of $[t_0, \infty)$. But this is a direct consequence of the Lebesgue dominated convergence theorem. Since all the assumptions of the Schauder-Tychonoff fixed point theorem are fulfilled, we obtain the existence of at least one fixed point $\mathbf{x} \in \Omega$ of the operator $\mathbf{T}$. This fixed point $\mathbf{x} = (x_1, \dots, x_n)$ is a positive solution of (4.77), and from the definition of the set Ω it follows that $x_i(t) \to 0$ as $t \to \infty$, $i = 1, \dots, n$, i.e., $\mathbf{x} \in \mathcal{SDS}$.

Now we prove that $x_i \in \mathcal{RV}(\nu_i)$, $i = 1, \dots, n$. Since $\mathbf{x} \in \Omega$, we have $x_i(t) \asymp t^{\nu_i}\tilde{L}_i(t)$ as $t \to \infty$, $i = 1, \dots, n$. Taking into account that $\tilde{L}_i(\lambda t)/\tilde{L}_i(t) \to 1$ as $t \to \infty$ for every $\lambda > 0$, $i = 1, \dots, n$, we can find $m_i, M_i \in (0, \infty)$, $i = 1, \dots, n$, such that

$$m_i \leq \tau_i(t) \leq M_i, \text{ where } \tau_i(t) := \frac{x_i(\lambda t)}{x_i(t)}, \quad i = 1, \dots, n, \tag{4.114}$$

for $t \geq t_0$, and so

$$\liminf_{t\to\infty} \tau_i(t) =: \underline{\Lambda}_i \in (0, \infty), \quad \limsup_{t\to\infty} \tau_i(t) =: \overline{\Lambda}_i \in (0, \infty).$$

From the uniform convergence theorem for $\mathcal{SV}_0$ functions, we get

$$\begin{aligned} \left| \frac{L_{F_i}(x_{i+1}(\lambda t))}{L_{F_i}(x_{i+1}(t))} - 1 \right| &= \left| \frac{L_{F_i}(\tau_{i+1}(t)x_{i+1}(t))}{L_{F_i}(x_{i+1}(t))} - 1 \right| \\ &\leq \sup_{\xi\in[m_i,M_i]} \left| \frac{L_{F_i}(\xi x_{i+1}(t))}{L_{F_i}(x_{i+1}(t))} - 1 \right| = o(1) \end{aligned}$$

as $t \to \infty$. Thus,

$$\begin{aligned}
\underline{\Lambda}_i &\geq \liminf_{t\to\infty} \frac{\lambda x_i'(\lambda t)}{x_i'(t)} = \liminf_{t\to\infty} \frac{\lambda a_i(\lambda t) x_{i+1}^{\alpha_i}(\lambda t)}{a_i(t) x_{i+1}^{\alpha_i}(t)} \cdot \frac{L_{F_i}(x_{i+1}(\lambda t))}{L_{F_i}(x_{i+1}(t))} \\
&\geq \lambda^{1+\sigma_i} \left(\liminf_{t\to\infty} \frac{x_{i+1}(\lambda t)}{x_{i+1}(t)} \right)^{\alpha_i} \geq \lambda^{1+\sigma_i} \left(\liminf_{t\to\infty} \frac{\lambda x_{i+1}'(\lambda t)}{x_{i+1}'(t)} \right)^{\alpha_i} \\
&= \lambda^{1+\sigma_i} \left(\liminf_{t\to\infty} \frac{\lambda a_{i+1}(\lambda t) x_{i+2}^{\alpha_{i+1}}(\lambda t)}{a_{i+1}(t) x_{i+2}^{\alpha_{i+1}}(t)} \cdot \frac{L_{F_{i+1}}(x_{i+2}(\lambda t))}{L_{F_{i+1}}(x_{i+2}(t))} \right)^{\alpha_i} \\
&\geq \lambda^{1+\sigma_i+\alpha_i(1+\sigma_{i+1})} \left(\liminf_{t\to\infty} \frac{\lambda x_{i+2}'(\lambda t)}{x_{i+2}'(t)} \right)^{\alpha_i \alpha_{i+1}} \\
&\geq \cdots \\
&\geq \lambda^{1+\sigma_i + A_{i,i+1}(1+\sigma_{i+1}) + A_{i,i+2}(1+\sigma_{i+2}) + \cdots + A_{i,i+n-1}(1+\sigma_{i+n-1})} \left(\liminf_{t\to\infty} \tau_{i+n}(t) \right)^{A_{i,i+n}} \\
&= \lambda^{\nu_i(1-A_{i,i+n})} \underline{\Lambda}_i^{A_{i,i+n}},
\end{aligned} \tag{4.115}$$

where we used (4.91). Realizing now that $A_{i,i+n} = \alpha_1 \cdots \alpha_n < 1$, we obtain $\underline{\Lambda}_i \geq \lambda^{\nu_i}$. Similarly we get $\overline{\Lambda}_i \leq \lambda^{\nu_i}$. This implies that there exists the limit $\lim_{t\to\infty} x(\lambda t)/x(t)$ and it is equal to λ^{ν_i}. Since λ was arbitrary, we get $x_i \in \mathcal{RV}(\nu_i)$, $i = 1, \ldots, n$.

Finally, we establish asymptotic formula (4.97). We have $x_i(t) = t^{\nu_i} \bar{L}_i(t)$, $i = 1, \ldots, n$, where $\bar{L}_i \in \mathcal{SV}$ has to be determined. Then, taking into account (4.89), (4.95), and (1.8), it results

$$\begin{aligned}
t^{\nu_i} \bar{L}_i(t) &= \int_t^\infty a_i(s) F_i(x_{i+1}(s))\, \mathrm{d}s = \int_t^\infty s^{\sigma_i} L_{a_i}(s) x_{i+1}^{\alpha_i}(s) L_{F_i}(x_{i+1}(s))\, \mathrm{d}s \\
&= \int_t^\infty s^{\nu_i - 1} L_{a_i}(s) \bar{L}_{i+1}^{\alpha_i}(s) L_{F_i}\left(s^{\nu_{i+1}} \bar{L}_{i+1}(s)\right) \mathrm{d}s \\
&\sim \int_t^\infty s^{\nu_i - 1} L_{a_i}(s) \bar{L}_{i+1}^{\alpha_i}(s) L_{F_i}\left(s^{\nu_{i+1}}\right) \mathrm{d}s \\
&\sim \frac{1}{|\nu_i|} t^{\nu_i} L_{a_i}(t) \bar{L}_{i+1}^{\alpha_i}(t) L_{F_i}\left(t^{\nu_{i+1}}\right)
\end{aligned}$$

as $t \to \infty$, $i = 1, \ldots, n$. Hence, $\bar{L}_i(t) \sim \tilde{L}_i(t) = h_i L_i(t)$, $i = 1, \ldots, n$, see Lemma 4.4, and (4.97) follows.

The proof of (ii) is similar and hence omitted. Just note that relation (1.9) finds its extensive application here. ◻

The following lemma, which follows the well known generalized AM-GM inequality, is needed to prove Theorem 4.15.

Lemma 4.10. *Let $u_1, \ldots, u_n > 0$ and $p_1, \ldots, p_n \geq 0$ with $p_1 + \cdots + p_n = 1$. Then*

$$\sum_{i=1}^n u_i \geq \prod_{i=1}^n u_i^{p_i}. \tag{4.116}$$

Proof of Theorem 4.15. (i) Take any $(x_1,\ldots,x_n)\in\mathcal{SDS}$, which indeed exists by Theorem 4.14. From (4.77), x_i satisfies the integral equation

$$x_i(t)=\int_t^\infty a_i(s)x_{i+1}^{\alpha_i}(s)\,\mathrm{d}s, \tag{4.117}$$

$i=1,\ldots,n$. From Lemma 4.8, there exists $m\in\{1,2,\ldots,n\}$ such that $\varrho_{i,m}<0$ for all $i=1,\ldots,n$, where the constants $\varrho_{i,j}$ are defined by (4.105) and satisfy also (4.106). Iterating (4.117), starting from $i=m$ we get

$$\begin{aligned}
x_m(t)&=\int_t^\infty a_m(s_1)x_{m+1}^{\alpha_m}(s_1)\,\mathrm{d}s_1\\
&=\int_t^\infty a_m(s_1)\left(\int_{s_1}^\infty a_{m+1}(s_2)x_{m+2}^{\alpha_{m+1}}\mathrm{d}s_2\right)^{\alpha_m}\mathrm{d}s_1\\
&=\cdots=\int_t^\infty a_m(s_1)\left(\int_{s_1}^\infty a_{m+1}(s_2)\times\right.\\
&\quad\left.\times\left(\ldots\left(\int_{s_{n-1}}^\infty a_{m+n-1}(s_n)x_{m+n}^{\alpha_{m+n-1}}(s_n)\,\mathrm{d}s_n\right)^{\alpha_{m+n-2}}\ldots\right)^{\alpha_{m+1}}\mathrm{d}s_2\right)^{\alpha_m}\mathrm{d}s_1
\end{aligned}$$

Since $x_{m+n}=x_m$ is eventually decreasing, $x_{m+n}(s_n)\le x_m(t)$ for $s_n\ge t$, t being sufficiently large. Further, $a_{m+n-1}(t)=t^{\sigma_{m+n-1}}L_{a_{m+n-1}}(t)$, with $\sigma_{m+n-1}=\varrho_{m+n-1,m}-1<-1$, see (4.106) and Lemma 4.8. Thus we can apply (1.8) obtaining the existence of a positive constant k_n such that

$$\int_{s_{n-1}}^\infty a_{m+n-1}(s_n)x_{m+n}^{\alpha_{m+n-1}}(s_n)\,\mathrm{d}s_n\le k_n x_m(s_{n-1})s_{n-1}^{\sigma_{m+n}}L_{a_{m+n-1}}(s_{n-1})$$

We can proceed in a similar way for all the iterated integrals; note that (4.106) and Lemma 4.8 assure that we can apply (1.8) at each step. We obtain that $k\in(0,\infty)$ exists such that for t large

$$\begin{aligned}
x_m(t)&\le x_m^{\alpha_m\cdots\alpha_{m+n-1}}(t)\times\\
&\quad\times\int_t^\infty s_1^{\sigma_m}L_{a_m}(s_1)\left(\ldots\left(\int_{s_{n_1}}^\infty s_n^{\sigma_{m+n-1}}L_{a_{m+n-1}}(s_n)\,\mathrm{d}s_n\right)^{\alpha_{m+n-2}}\ldots\right)^{\alpha_m}\mathrm{d}s_1\\
&\le kx_m^{A_{m,m+n}}L_{a_m}(t)L_{a_{m+1}}^{A_{m,m+1}}(t)\cdots L_{a_{m+n-1}}^{A_{m,m+n-1}}(t)t^{\nu_m(1-A_{m,m+n})}
\end{aligned}$$

where we used the equality $\nu_m(1-A_{m,m+n})=\sigma_m+1+A_{m,m+1}(\sigma_{m+1}+1)+\cdots+A_{m,m+n-1}(\sigma_{m+n-1}+1)$, see (4.91). Since $A_{m,m+n}=\alpha_1\cdots\alpha_n<1$, from (4.92) with $L_{F_1}\equiv\cdots\equiv L_{F_n}\equiv 1$, there exists $d_m\in(0,\infty)$ such that $x_m(t)\le d_m t^{\nu_m}L_m(t)$ for large t.

Now we show that $x_i(t)\le d_i t^{\nu_i}L_i(t)$ for large t and for all $i=1,\ldots,n$, with $d_i>0$. From the estimate for $x_m=x_{m+n}$, we can now easily get the estimation for x_{m+n-1}. Recall that (4.89) and (4.94) hold. From (4.117) and (1.8), in view of $\nu_m<0$, we have

$$
\begin{aligned}
x_{m+n-1}(t) &= \int_t^\infty a_{m+n-1}(s) x_m^{\alpha_{m+n-1}}(s)\,\mathrm{d}s \\
&\le d_m^{\alpha_{m+n-1}} \int_t^\infty s^{\sigma_{m+n-1}+\alpha_{m+n-1}\nu_m} L_{a_{m+n-1}}(s) L_m^{\alpha_{m+n-1}}(s)\,ds \\
&\le d_{m+n-1} t^{\nu_{m+n-1}} L_{a_{m+n-1}}(t) L_m^{\alpha_{m+n-1}}(t) = d_{m+n-1} t^{\nu_{m+n-1}} L_{m+n-1}(t)
\end{aligned}
$$

for large t, where d_{m+n-1} is a suitable positive constant. Repeating this process, and taking into account the modulo n convention, it results $x_i(t) \le d_i t^{\nu_i} L_i(t)$ for large t and for all $i = 1, \dots, n$.

Next we derive lower estimates for x_i's. For brevity we sometimes omit arguments. Again take any $(x_1, \dots, x_n) \in \mathcal{SDS}$. Then

$$
\begin{aligned}
-(x_1 x_2 \cdots x_n)' &= -\sum_{i=1}^n x_i' x_{i+1} \cdots x_{i+n-1} = \sum_{i=1}^n a_i x_{i+1}^{\alpha_i} x_{i+1} \cdots x_{i+n-1} \\
&= \sum_{i=1}^n H_i, \quad \text{where } H_i = a_i x_{i+1}^{\alpha_i} \prod_{k=1;k\ne i}^n x_k.
\end{aligned}
$$

Consider the (positive) numbers $p_1, \dots, p_n$ defined in Lemma 4.5. System (4.101) is equivalent to the system

$$
\begin{cases}
p_1 + \cdots + p_n = 1, \\
p_2 + p_3 + \cdots + p_{n-1} + p_n(\alpha_n + 1) = p_1(\alpha_1 + 1) + p_3 + \cdots + p_n, \\
p_1(\alpha_1 + 1) + p_3 + \cdots + p_n = p_1 + p_2(\alpha_2 + 1) + p_4 + \cdots + p_n, \\
\quad \vdots \\
p_1 + \cdots + p_{n-3} + p_{n-2}(\alpha_{n-2} + 1) + p_n = p_1 + \cdots + p_{n-2} + p_{n-1}(\alpha_{n-1} + 1).
\end{cases}
\tag{4.118}
$$

From Lemma 4.10 we get that

$$
\begin{aligned}
-(x_1 x_2 \cdots x_n)' &= \sum_{i=1}^n H_i \ge \prod_{i=1}^n H_i^{p_i} \\
&= a_1^{p_1} a_2^{p_2} \cdots a_n^{p_n} x_1^{p_2+p_3+\cdots+p_{n-1}+p_n(\alpha_n+1)} x_2^{p_1(\alpha_1+1)+p_3+\cdots+p_n} \times \\
&\quad \times \cdots x_{n-1}^{p_1+p_2+\cdots+p_{n-2}(\alpha_{n-2}+1)+p_n} x_n^{p_1+p_2+\cdots+p_{n-2}+p_{n-1}(\alpha_{n-1}+1)}
\end{aligned}
$$

Observe that except of the first equality all sides of the equalities in (4.118) are mutually equal and denote any of them by ξ; this is the same ξ as in Lemma 4.5. Then, from the last estimate, we get

$$
-(x_1 \cdots x_n)' \ge a_1^{p_1} \cdots a_n^{p_n} (x_1 \cdots x_n)^{\xi}. \tag{4.119}
$$

By Lemma 4.5, we have $\xi < 1$. Dividing (4.119) by $(x_1 \cdots x_n)^\xi$ and integrating from t to ∞, we obtain

$$(x_1(t) \cdots x_n(t))^{1-\xi} \geq (1-\xi) \int_t^\infty a_1^{p_1}(s) \cdots a_n^{p_n}(s)\, \mathrm{d}s. \tag{4.120}$$

From the upper estimates for x_i's we have

$$x_1(t) \cdots x_n(t) \leq l_1 x_i(t) t^{\sum_{k=1;k\neq i}^n \nu_k} \prod_{k=1;k\neq i}^n L_k(t) \tag{4.121}$$

for large t, where $i \in \{1, \ldots, n\}$ and l_1 is some positive number. Taking into account that $\sum_{i=1}^n \sigma_i p_i < -1$, see (4.103), from (1.8) there exists $l_2 \in (0, \infty)$ such that

$$\int_t^\infty a_1^{p_1}(s) \cdots a_n^{p_n}(s)\, \mathrm{d}s \geq l_2 t^{\sigma_1 p_1 + \cdots + \sigma_n p_n + 1} L_{a_1}^{p_1}(t) \cdots L_{a_n}^{p_n}(t) \tag{4.122}$$

for large t. Combining (4.120), (4.121), and (4.122), we find $c_i \in (0, \infty)$ such that

$$x_i(t) \geq c_i t^{\bar{\nu}_i} \bar{L}_i(t)$$

for large t, where

$$\bar{\nu}_i = \frac{1}{1-\xi}(\sigma_1 p_1 + \cdots \sigma_n p_n + 1) - \sum_{k=1;k\neq i}^n \nu_k \quad \text{and} \quad \bar{L}_i = \frac{\left(L_{a_1}^{p_1} \cdots L_{a_n}^{p_n}\right)^{\frac{1}{1-\xi}}}{\prod_{k=1;k\neq i}^n L_k}$$

Identities (4.103) and (4.104) now imply $\bar{\nu}_i = \nu_i$ and $\bar{L}_i(t) = L_i(t)$, $i = 1, \ldots, n$. Thus we have proved $x_i(t) \asymp t^{\nu_i} L_i(t)$ as $t \to \infty$, $i = 1, \ldots, n$. This implies (4.114). The property $x_i \in \mathcal{RV}(\nu_i)$, $i = 1, \ldots, n$, and asymptotic formula (4.97) follow by the same arguments as those used in the second part of the proof of Theorem 4.14-(i).

(ii) The proof of this statement can be found in [146], and uses similar arguments as part (i). Note that certain additional condition is assumed in [146]. But in view of Lemma 4.8, under the assumptions of Theorem 4.15-(ii), that additional condition is automatically satisfied. $\square$

Theorem 4.15 can be applied, for instance, to the equations

$$x^{(n)} = (-1)^n p(t) \Phi_\beta(x), \tag{4.123}$$

and

$$x^{(n)} = p(t) \Phi_\beta(x), \tag{4.124}$$

where $p(t) = t^\varrho L_p(t)$, $L_p \in \mathcal{SV}$ and $0 < \beta < 1$, leading to the following results.

(a) If $\varrho + n < 0$, then (4.123) possesses a solution x such that $\lim_{t\to\infty} x^{(i)}(t) = 0$, $i = 0, \ldots, n-1$, and for any such a solution it holds

$$x \in \mathcal{RV}\left(\frac{\varrho + n}{1-\beta}\right) \tag{4.125}$$

with

$$x^{1-\beta}(t) \sim t^{\varrho+n} L_p(t) \prod_{j=1}^{n} \frac{1-\beta}{-\varrho - n + (1-\beta)(j-1)} \text{ as } t \to \infty.$$

(b) If $\varrho + 1 + \beta(n-1) > 0$, then equation (4.124) possesses a solution x such that $\lim_{t\to\infty} x^{(i)}(t) = \infty$, $i = 1, \dots, n-1$, and for any such a solution (4.125) holds with

$$x^{1-\beta}(t) \sim t^{\varrho+n} L_p(t) \prod_{j=1}^{n} \frac{1-\beta}{\varrho + n - (1-\beta)(j-1)} \text{ as } t \to \infty.$$

The theorems of this subsection can clearly be applied to the equation of the form (4.82), or even to the more general equation

$$D_{q_n}(\gamma_n) D_{q_{n-1}}(\gamma_{n-1}) \cdots D_{q_1}(\gamma_1) x + \delta p(t) \Phi_\beta(x) = 0 \tag{4.126}$$

with $D_f(\gamma)x = \frac{d}{dt}(f(t)\Phi_\gamma(x))$. As a special case of (4.126), we get the (above discussed) equation

$$(r(t)\Phi_\alpha(x'))' = p(t)\Phi_\beta(x). \tag{4.127}$$

We point out that equations of this form (but this holds also for some higher order equations or for second order systems), are often studied in separate settings depending on whether the integral $\int_a^\infty r^{-1/\alpha}(s)\,\mathrm{d}s$ diverges or not. The results of this subsection are in such a form that this distinction is not necessary and all possible combinations which may occur in second order systems or higher order equations are included in our setting.

In Subsection 5.1.2 we indicate another application of Theorem 4.14, it concerns a scalar second order equation with generalized Φ-Laplacian.

Chapter 5

Some other nonlinear differential equations

5.1 Equations with a general Φ-Laplacian

As a typical prototype of the objects from the title of this section, let us consider the second order equation

$$(r(t)G(y'))' = p(t)F(y), \tag{5.1}$$

where $r > 0, p$ are continuous functions on $[a, \infty)$, G is an increasing odd homeomorphism defined on an open interval $(-\rho, \rho)$, $0 < \rho \le \infty$, and $\operatorname{Im} G = (-\sigma, \sigma)$, $0 < \sigma \le \infty$, F is a real continuous function on $\mathbb{R}$ such that $uF(u) > 0$ for $u \ne 0$.

In fact, Emden-Fowler type equations (which are studied in the previous chapter) are sometimes in so general form that include also equation (5.1). For instance, the general setting of Subsection 4.3.8 allows us to apply (after a modification) Theorem 4.14 to equation (5.1). This fact will be illustrated in Subsection 5.1.2, where we examine certain boundary value problem involving equation (5.1).

Note that typically — in considerations within the theory in the framework of regular variation — it is assumed that G^{-1} or a generalized inverse of G is regularly varying. Examples satisfying these assumptions are the classical p-Laplacian

$$G(u) = \Phi_\alpha(u) = |u|^\alpha \operatorname{sgn} u \in \mathcal{RV}(\alpha) \cap \mathcal{RV}_0(\alpha)$$

(even this function is trivially regularly varying), or

$$G(u) = \Phi_\alpha(u) \ln |u| \in \mathcal{RV}(\alpha) \cap \mathcal{RV}_0(\alpha),$$

or

$$G(u) = u^\delta (A + Bu^\beta)^\gamma \in \mathcal{RV}(\delta + \beta\gamma) \cap \mathcal{RV}_0(\delta)$$

if $\delta, \beta, \gamma > 0$. Other, more special and typical, prototypes are $G(u) = \Phi_C(u)$ or $G(u) = \Phi_R(u)$, where

$$\Phi_C(u) = \frac{u}{\sqrt{1+u^2}} \text{ and } \Phi_R(u) = \frac{u}{\sqrt{1-u^2}},$$

these operators arise in studying radially symmetric solutions of partial differential equations with the mean curvature operator and the relativity operator, respectively. Note that $\Phi_C^{-1} = \Phi_R$, $\Phi_R^{-1} = \Phi_C$, and $\Phi_C, \Phi_R \in \mathcal{RV}_0(1)$.

Recall that the inverse of an increasing regularly varying function of index $\vartheta > 0$ is in $\mathcal{RV}(1/\vartheta)$. A similar statement can be proved for functions in $\mathcal{RV}_0(\vartheta)$.

In some considerations, thanks to the theorem about asymptotic inversions of $\mathcal{RV}$ functions, the assumption on the monotonicity of G can be omitted; one can then work, for instance, with the generalized inverse instead of the inverse.

In working with some forms of G, like, for instance, $G(x) = x \ln |x|$, the Lambert W function may play an important role. The inverse of G is in this case $G^{-1}(x) = e^{W(x)}$. Note that W cannot be expressed in terms of elementary functions.

5.1.1 A boundary value problem on a half-line

The results of this subsection are based on the paper [27] by Došlá, Marini, and Matucci. We are interested in solving the BVP on the whole half-line associated to (5.1), $r(t) > 0, p(t)$ being defined for $t \in [0, \infty)$, especially when the weight p changes its sign, that is, if there exist $t_1, t_2 \geq 0$ satisfying $p(t_1)p(t_2) < 0$. In addition to the hypotheses at (5.1), it is assumed that F is nondecreasing and $\liminf_{t\to\infty} r(t) > 0$. The boundary conditions read as

$$y(0) = c > 0, \quad y(t) > 0, \quad 0 < \lim_{t\to\infty} y(t) < \infty, \quad \lim_{t\to\infty} y'(t) = 0. \tag{5.2}$$

Let p_+, p_-, denote respectively the positive and the negative part of p. Clearly, $p(t) = p_+(t) - p_-(t)$.

All the cases when

the inverse G^{-1} of G is $\mathcal{RV}_0$ or $\mathcal{RPV}_0$ or $\mathcal{SV}_0$

are discussed in [27]. For illustration, we present one selected result and the shortened proof. A crucial role in all the proofs is played by the Schauder-Tychonoff fixed point theorem. Very helpful are also various facts which are consequences of simple properties of $\mathcal{RV}$ and $\mathcal{RPV}$ functions like, for instance, $g(u) \leq Mu^{\alpha-\varepsilon}$, $0 < \varepsilon < \alpha$, or $g(\lambda u) \leq M\lambda^\alpha g(u)$, $\lambda \in (0, 1]$, on $(0, T]$ for some $M > 0$, provided $g \in \mathcal{RV}_0(\alpha)$, $\alpha > 0$, etc.

Theorem 5.1. *Let $G^{-1} \in \mathcal{RV}_0(\alpha)$ with $\alpha > 0$, and assume that*

$$\lim_{u\to0+} \frac{F(u)}{u^{1/\beta}} = 0, \tag{5.3}$$

and for some $\varepsilon \in (0, \alpha - \beta)$*,*

$$\begin{aligned}
&\int_0^\infty \left(\frac{1}{r(s)} \int_s^\infty p_+(\tau)\,\mathrm{d}\tau \right)^{\alpha-\varepsilon} \mathrm{d}s < \infty, \\
&\int_0^\infty \left(\frac{1}{r(s)} \int_s^\infty p_-(\tau)\,\mathrm{d}\tau \right)^{\alpha-\varepsilon} \mathrm{d}s < \infty.
\end{aligned} \tag{5.4}$$

Then the boundary value problem (5.1), (5.2) *is solvable for any small positive* c*. Moreover, every solution is of bounded variation on* $[0, \infty)$*.*

Proof. Choose $\mu > 0$ such that

$$T_\mu = \mu \max \left\{ \max_{t \ge 0} \left(\frac{1}{r(t)} \int_t^\infty p_+(s)\,\mathrm{d}s \right), \min_{t \ge 0} \left(\frac{1}{r(t)} \int_t^\infty p_-(s)\,\mathrm{d}s \right) \right\} < \sigma.$$

Since $G^{-1} \in \mathcal{RV}_0(\alpha)$, having a fixed $\varepsilon < \alpha - \beta$, a positive constant M exists such that $G^{-1}(u) \le M u^{\alpha-\varepsilon}$ for $0 < u \le T_\mu$. Choose $c > 0$ sufficiently small such that $F(2c) \le \mu$. Let Ω be the subset of the Fréchet space $C[0, \infty)$ given by

$$\Omega = \left\{ u \in C[0, \infty) : \frac{c}{2} \le u(t) \le 2c \right\}$$

and define in Ω the operator $\mathcal{T}$ as follows

$$\mathcal{T}(u)(t) = c + \int_0^t G^{-1} \left(\frac{1}{r(s)} \left(\int_s^\infty p_-(\tau) F(u(\tau))\,\mathrm{d}\tau - \int_s^\infty p_+(\tau) F(u(\tau))\,\mathrm{d}\tau \right) \right).$$

By the previous considerations, $\mathcal{T}$ is well defined. It can be shown that $\mathcal{T}(\Omega) \subset \Omega$, $\mathcal{T}(\Omega)$ is relative compact, and $\mathcal{T}$ is continuous. Thus we can apply the Schauder-Tychonoff fixed point theorem, which guarantees the existence of $y \in \Omega$ such that $y(t) = \mathcal{T}(y)(t)$, i.e. y is a solutions of (5.1). Clearly,

$$y'(t) = G^{-1} \left(\frac{1}{r(t)} \left(\int_t^\infty p_-(s) F(y(s))\,\mathrm{d}s - \int_t^\infty p_+(s) F(y(s))\,\mathrm{d}s \right) \right)$$

and so

$$-G^{-1} \left(\frac{F(2c)}{r(t)} \int_t^\infty p_+(s)\,\mathrm{d}s \right) \le y'(t) \le G^{-1} \left(\frac{F(2c)}{r(t)} \int_t^\infty p_-(s)\,\mathrm{d}s \right).$$

Since $1/r(t)$ is bounded as $t \to \infty$, we have $\lim_{t\to\infty} y'(t) = 0$ and, from (5.4), $y' \in L^1[0, \infty)$. Thus, y is of bounded variation on $[0, \infty)$ and the limit $\lim_{t\to\infty} y(t)$ is finite. Since y belongs to Ω, the assertion follows. □

Note that condition (5.3) is satisfied, for instance, if $F \in \mathcal{RV}_0(\gamma)$, where $\gamma > 1/\beta$.

5.1.2 Regularly varying solutions

Theorem 4.14 can be modified, for instance, for a system of the form

$$x_i' = a_i(t)G_i(b_i(t), x_{i+1}),$$

$i = 1, \dots, n$, where x_{n+1} means x_1, a_i, b_i are regularly varying, and G_i are regularly varying with respect to both variables. Such a modification enables us to include equations like (5.1), where r, p are positive continuous functions on $[a, \infty)$ with $r \in \mathcal{RV}(\sigma)$, $p \in \mathcal{RV}(\varrho)$, and F, G are continuous functions on $\mathbb{R}$ with $uF(u) > 0$, $uG(u) > 0$ for $u \neq 0$, $|G(|\cdot|)| \in \mathcal{RV}_0(\alpha)$, $|F(|\cdot|)| \in \mathcal{RV}_0(\beta)$, $\alpha, \beta \in (0, \infty)$, G being increasing in a neighborhood of zero. In fact, the theorem about asymptotic inversions of $\mathcal{RV}$ allows us to omit the assumption on the monotonicity of G, but for simplicity we assume it. Equation (5.1) can be written as

$$\begin{cases} x_1' = p(t)F(x_2), \\ x_2' = G^{-1}\left(\frac{1}{r(t)}x_1\right), \end{cases}$$

where G^{-1} stand for the inverse of G; it holds $G^{-1} \in \mathcal{RV}_0(1/\alpha)$. Assume that

$$L_F(ug(u)) \sim L_F(u), \quad L_{G^{-1}}(ug(u)) \sim L_{G^{-1}}(u) \quad \text{as } u \to 0+ \tag{5.5}$$

for every $g \in \mathcal{SV}_0$. The subhomogeneity condition reads as $\alpha > \beta$. If

$$\varrho + 1 < \min\left\{\sigma - \alpha, \frac{\beta}{\alpha}(\sigma - \alpha)\right\},$$

then (5.1) possesses an eventually positive decreasing solution x such that

$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty} r(t)G(x'(t)) = 0, \tag{5.6}$$

$x \in \mathcal{RV}(\nu)$, and

$$x^{\alpha-\beta}(t) \sim \frac{1}{-(\varrho + 1 + \beta\nu)(-\nu)^\alpha} \cdot \frac{L_{G^{-1}}^\alpha(t^{\varrho+1+\beta\nu-\sigma})L_F(t^\nu)L_p(t)}{L_r(t)} \cdot t^{\nu(\alpha-\beta)} \tag{5.7}$$

as $t \to \infty$, where $\nu = (\alpha - \sigma + \varrho + 1)/(\alpha - \beta)$. Note that also $x'(t)$ tends to zero as $t \to \infty$. For completeness we add that if, in addition, $F = \Phi_\beta$ and $G = \Phi_\alpha$, then every solution x of (5.1) with (5.6) then satisfies $x \in \mathcal{RV}(\nu)$ and (5.7). We already know that other examples of G, different from the classical p-Laplacian case, are $G_C(u)$ and $G_R(u)$, the *curvature operator* and the *relativity operator*, respectively, both are in $\mathcal{RV}_0(1)$, and are mutually inverse. Note that condition (5.5) is clearly fulfilled for G_C, G_R.

5.2 Partial differential equations

5.2.1 Radial $\mathcal{RV}$ solutions of partial differential systems

It is clear that many of the results in this text (that are established for ordinary differential equations) give useful information about asymptotic form of radial solutions to associated partial differential equations. For example, let us show how Theorem 4.11 can be applied to the partial differential system

$$\begin{cases} \operatorname{div}\left(\|\nabla u\|^{\alpha-1}\nabla u\right) = \bar{\varphi}(\|z\|)|v|^{\lambda-1}v, \\ \operatorname{div}\left(\|\nabla v\|^{\beta-1}\nabla v\right) = \bar{\psi}(\|z\|)|u|^{\mu-1}u \end{cases} \tag{5.8}$$

in an exterior domain in $\mathbb{R}^N$, $N \geq 2$, where $\bar{\varphi}(t) = t^{\bar{\sigma}} L_{\bar{\varphi}}(t) \in \mathcal{RV}(\bar{\sigma})$, $\bar{\psi}(t) = t^{\bar{\varrho}} L_{\bar{\psi}}(t) \in \mathcal{RV}(\bar{\varrho})$. If we assume

$$\begin{aligned} &\bar{\nu} := \Lambda(\beta(\alpha + 1 + \bar{\sigma}) + \lambda(\beta + 1 + \bar{\varrho})) < 0, \\ &\bar{\omega} := \Lambda(\alpha(\beta + 1 + \bar{\varrho}) + \mu(\alpha + 1 + \bar{\sigma})) < 0, \\ &N < \min\{1 - (\bar{\nu} - 1)\alpha, 1 - (\bar{\omega} - 1)\beta\}, \end{aligned}$$

then the existence of a positive (strongly) decreasing radial solution of (5.8) is guaranteed, and any such a solution (u, v) satisfies

$$\begin{aligned} &\lim_{\|z\|\to\infty} \frac{u(z)}{|z|^{\bar{\nu}} L_{\bar{\varphi}}^{\beta\Lambda}(\|z\|) L_{\bar{\psi}}^{\lambda\Lambda}(\|z\|)} = \left(\bar{K}_1 \bar{K}_2^{\frac{\lambda}{\alpha}}\right)^{\alpha\beta\Lambda}, \\ &\lim_{\|z\|\to\infty} \frac{v(z)}{|z|^{\bar{\omega}} L_{\bar{\varphi}}^{\mu\Lambda}(\|z\|) L_{\bar{\psi}}^{\alpha\Lambda}(\|z\|)} = \left(\bar{K}_2 \bar{K}_1^{\frac{\mu}{\beta}}\right)^{\alpha\beta\Lambda}, \end{aligned}$$

where $\bar{K}_1 = -\left(\bar{\nu}\left((1-\bar{\nu})\alpha - N + 1\right)^{\frac{1}{\alpha}}\right)^{-1}$, $\bar{K}_2 = -\left(\bar{\omega}\left((1-\bar{\omega})\beta - N + 1\right)^{\frac{1}{\beta}}\right)^{-1}$. Indeed, a radial function $(u(z), v(z))$ is a solution of (5.8) in $\Sigma_a = \{z \in \mathbb{R}^N : \|z\| \geq a\}$ if and only if $(x(t), y(t))$, $t = \|z\|$, given by $(x(\|z\|), y(\|z\|)) = (u(z), v(z))$, satisfies the ordinary differential system

$$\begin{cases} \left(t^{N-1}\Phi_\alpha(x')\right)' = t^{N-1}\bar{\varphi}(t)\Phi_\lambda(y), \\ \left(t^{N-1}\Phi_\beta(y')\right)' = t^{N-1}\bar{\psi}(t)\Phi_\mu(x), \end{cases} \tag{5.9}$$

$t \geq a$. System (5.9) has the same structure as (4.40), with $p(t) = q(t) = t^{N-1}$, $\varphi(t) = t^{N-1}\bar{\varphi}(t)$, $\psi(t) = t^{N-1}\bar{\psi}(t)$. Thus $\gamma = \delta = N-1$, $L_p(t) = L_q(t) = 1$, $\sigma = N - 1 + \bar{\sigma}$, $\varrho = N - 1 + \bar{\varrho}$, $L_\varphi(t) = L_{\bar{\varphi}}(t)$, $L_\psi(t) = L_{\bar{\psi}}(t)$. Applying Theorem 4.11 to system (5.9) and going back to the original variables we get the result.

Note that if Theorem 4.11 was obtained just for system (4.40) with $p(t) = q(t) = 1$, then, after a suitable transformation of the independent variable, only partial systems of the form (5.8) satisfying the conditions $\alpha = \beta$ and $\alpha + 1 > N$ would be detectable. Arbitrariness of p, q and of the convergence or divergence of the integrals in (4.41) enables us to omit these restrictions. Moreover, our general setting allows us to consider even more general partial differential systems where the leading coefficients are formed by elliptic matrices of certain special forms.

Similarly we can apply Theorem 4.14 and Theorem 4.15 when studying positive radial solutions to partial differential systems of the form

$$\begin{cases} \operatorname{div}(\|\nabla u_1\|^{\lambda_1-1}\nabla u_1) & = \varphi_1(\|z\|)G(u_2), \\ \operatorname{div}(\|\nabla u_2\|^{\lambda_2-1}\nabla u_2) & = \varphi_2(\|z\|)G(u_3), \\ & \vdots \\ \operatorname{div}(\|\nabla u_k\|^{\lambda_k-1}\nabla u_k) & = \varphi_k(\|z\|)G(u_1). \end{cases} \tag{5.10}$$

For some other information concerning related partial differential systems see e.g. [22, 24] and the references in [24, 127, 162]. Recall that systems similar to (5.8) are in the literature sometimes called as of *Lane-Emden type*. For information about applications of such systems see e.g. [169].

5.2.2 Almost radial symmetry

In this subsection we give another demonstration how $\mathcal{RV}$ functions can appear in the qualitative theory of differential equations. This time we consider the partial differential equation

$$\Delta u = \varphi(|x|)u^{\lambda}, \quad x \in \mathbb{R}^n \text{and } |x| \text{ large}, \tag{5.11}$$

where $n \geq 3$ is an integer, $\lambda \in (1, \infty)$ and, for some $t_0 > 0$, $\varphi : [t_0, \infty) \to (0, \infty)$ is a continuous function.

We present one commented result from Taliaferro [159] without a proof, and the interesting fact is to observe how the subject of regular variation is related to the conditions on φ in the theorem. It is worthy of note — as claimed by Taliaferro in the paper — that is was Omey who pointed him out this fact.

We are interested in whether all positive solutions of (5.11) are almost radial when $|x|$ is large, i.e.,

$$\frac{u(x)}{\bar{u}(|x|)} \to 1 \quad \text{as } |x| \to \infty,$$

where $\bar{u}(t)$ is the average of u on the sphere $|x| = t$. Assume that

$$\int_{t_0}^{\infty} s^{\alpha}\varphi(s)\,\mathrm{d}s < \infty \tag{5.12}$$

for some $\alpha > 1$ and note also other (mutually exclusive) cases are discussed in [159].

Theorem 5.2. *Suppose that $u(x)$ is a positive solution of* (5.11)*. Assume that*

$$\frac{\varphi(t)}{\psi(t)} \to 1 \quad \textit{as } t \to \infty \tag{5.13}$$

for some continuously differentiable function $\psi : [t_0, \infty) \to (0, \infty)$ *such that*

$$t^{2n-2}\psi(t) \text{ is monotone on } [t_0, \infty), \tag{5.14}$$

and

$$\lim_{t\to\infty}\left(\frac{1}{\sqrt{\psi(t)}}\right)' \int_t^\infty \sqrt{\psi(s)}\mathrm{d}s = M \text{ for some } M \in \mathbb{R} \cup \{\pm\}. \tag{5.15}$$

Then

$$\lim_{|x|\to\infty}\frac{u(x)}{\bar{u}(|c|)} = 1 \text{ and } \lim_{|x|\to\infty}[u(x) - \bar{u}(|x|)] = 0.$$

Moreover, $1 \le M \le (\alpha+1)/(\alpha-1)$ *and either*

$$\lim_{|x|\to\infty} u(x)\left(\frac{\lambda-2}{2}\int_{|x|}^\infty \sqrt{\psi(s)}\,\mathrm{d}s\right)^{\frac{2}{\lambda-1}} = \left(1 + \frac{(n-2)(\lambda-1)(M-1)}{2}\right)^{\frac{1}{\lambda-1}},$$

or

$$u(x) = c + o(1) \quad \text{as } |x| \to \infty,$$

or

$$u(x) = \frac{c+o(1)}{|x|^{n-2}} \quad \text{as } |x| \to \infty,$$

where c is some positive constant.

Remark 5.1. Let $\psi : [t_0, \infty) \to (0, \infty)$ be a continuously differentiable function. Using methods of the subject of regular variation one can easily show that ψ satisfies both $\int_{t_0}^\infty \sqrt{\psi(s)}\,\mathrm{d}s < \infty$ and (5.15) with $M = 1$ if and only if

$$\psi(t) = \psi(t_0)\exp\left\{-2\int_{t_0}^t \frac{1+h'(s)}{h(s)}\,\mathrm{d}s\right\}$$

for $t \ge t_0$ for some continuously differentiable function $h : [t_0, \infty) \to (0, \infty)$ such that $h'(t) \to 0$ as $t \to \infty$. In this case

$$\int_t^\infty \sqrt{\psi(s)}\,\mathrm{d}s = \left(\int_{t_0}^\infty \sqrt{\psi(s)}\,\mathrm{d}s\right)\exp\left\{-\int_{t_0}^t \frac{1}{h(s)}\,\mathrm{d}s\right)$$

for $t \ge t_0$ and for each $\beta \in \mathbb{R}$ there exists $t_1 > t_0$ such that $\psi(t)t^\beta$ is strictly decreasing on $[t_1, \infty)$. Therefore, since there are functions $h(t)$ as above which tend arbitrarily fast to zero as $t \to \infty$, we see that Theorem 5.2 allows $\psi(t)$ to tend arbitrarily fast to zero as $t \to \infty$ and allows solutions of (5.11) to tend arbitrarily fast to ∞ as $|x| \to \infty$. However conditions (5.13), (5.14), and (5.15) do require that $\psi(t)$ does not oscillate too much as $t \to \infty$.

Remark 5.2. Let $\varphi : [t_0, \infty) \to (0, \infty)$ be a continuous function such that

$$\int_{t_0}^{\infty} \sqrt{\psi(s)}\, \mathrm{d}s < \infty.$$

(As pointed out in the previous remark, this will be the case if (5.12) holds.) Then there exists a continuously differentiable function $\psi : [t_0, \infty) \to (0, \infty)$ satisfying both (5.13) and (5.15) for some $M \in (1, \infty)$ if and only if $\varphi \in \mathcal{RV}(-2\beta)$ for some $\beta \in (1, \infty)$. In this case $\beta M = \beta + M$

5.3 $\mathcal{RV}$ and $\mathcal{RPV}$ solutions for a class of third order nonlinear differential equations

The results of this section are based on the paper [65] by Jaroš, Kusano, and Marić. Let us consider the third order differential equation of the form

$$x''' + 2p(t)x' + p'(t)x = F(t, x), \tag{5.16}$$

where p is continuously differentiable on $[a, \infty)$, F is continuous on $[a, \infty) \times \mathbb{R}$ and

$$|F(t, u)| \le G(t, u)|,$$

where $G : [a, \infty) \times \mathbb{R} \to [0, \infty)$ is a continuous function which is nondecreasing in the second variable for $t \ge a$. Along with (5.16) consider the self-adjoint linear equation

$$y''' + 2p(t)y' + p'(t)y = 0. \tag{5.17}$$

With the help of the results for linear second order equations (Theorems 2.1, 2.2, 2.3, 2.4), we indicate the situation in which (5.17) has a fundamental set of solutions consisting of regularly and rapidly varying functions. We then establish the conditions under which (5.16) possesses a set of solutions which are asymptotic as $t \to \infty$ to the indicated $\mathcal{RV}$ and $\mathcal{RPV}$ solutions of (5.16).

5.3.1 The self-adjoint equation

Theorem 5.3. *Let C be a constant such that $C < 1/4$ and let ϱ, σ, with $\varrho < \sigma$, denote the real roots of $\lambda^2 - \lambda + C = 0$. If p is integrable on $[a, \infty)$ and satisfies*

$$\lim_{t\to\infty} t \int_t^{\infty} p(s)\, \mathrm{d}s = 2C, \tag{5.18}$$

then (5.17) *has a fundamental set of $\mathcal{RV}$ (hence nonoscillatory) solutions $y_i(t)$, $i = 1, 2, 3$, of the form*

$$y_1(t) = t^{2\varrho} L_1(t), \quad y_2(t) = tL_2(t), \quad y_3(t) = t^{2\sigma} L_3(t), \tag{5.19}$$

where $L_1 \in \mathcal{SV}$, $L_2(t) \to 1/(1 - 2\varrho)$, and $L_3(t) \sim 1/((1 - 2\varrho)^2 L_1(t))$ as $t \to \infty$.

Proof. By Theorems 2.2 and 2.3, condition (5.18) is necessary and sufficient for the equation

$$z'' + \frac{1}{2}p(t)z = 0 \tag{5.20}$$

to possess two linearly independent $\mathcal{RV}$ solutions of the form $u(t) = t^{\varrho}L_1(t)$ and $v(t) = t^{\sigma}L_2(t)$, where $L_1 \in \mathcal{SV}$ and $L_2(t) \sim 1/((1-2\varrho)^2 L_1(t))$ as $t \to \infty$. It is known that if $\{u, v\}$ is a fundamental set of solutions of (5.20), then $\{y_1, y_2, y_3\}$, where $y_1 = u^2, y_2 = uv, y_3 = v^2$, is a fundamental set of solutions of (5.17). Simple application of $\mathcal{RV}$ functions now gives the result. □

The borderline case ($C = 1/4$) between oscillation and nonoscillation of (5.20) is treated in the next theorem.

Theorem 5.4. *Suppose that*

$$\lim_{t\to\infty} \int_t^{\infty} p(s)\,\mathrm{d}s = \frac{1}{2}.$$

Suppose furthermore that the function

$$\phi(t) = t\int_t^{\infty} p(s)\,\mathrm{d}s - \frac{1}{2}$$

satisfies

$$\int^{\infty} \frac{|\phi(t)|}{t}\,\mathrm{d}t < \infty$$

and

$$\int^{\infty} \frac{|\psi(t)|}{t}\,\mathrm{d}t < \infty, \quad \textit{where} \quad \psi(t) = \int_t^{\infty} \frac{|\phi(s)|}{s}\,\mathrm{d}s < \infty.$$

Then equation (5.17) *has a fundamental set of* $\mathcal{RV}(1)$ *(hence nonoscillatory) solutions* $y_i(t)$*,* $i = 1, 2, 3$*, of the form*

$$y_1(t) = tL_1(t), \quad y_2(t) = t\ln tL_2(t), \quad y_3(t) = t\ln^3 tL_3(t),$$

where $L_1(t) \to k \in (0, \infty)$*,* $L_2(t) \to 1$*,* $L_3(t) \to 1/k^2$ *as* $t \to \infty$*.*

Proof. One proceeds exactly as in the proof of the previous theorem, using this time Theorem 2.4. □

Theorem 5.5. *Let* $p(t) < 0$ *for* $t \geq a$*. If for each* $\lambda > 1$

$$\lim_{t\to\infty}\left(-t\int_t^{\lambda t} p(s)\,\mathrm{d}s\right) = \infty, \tag{5.21}$$

then (5.17) *has at least two* $\mathcal{RPV}$ *solutions such that the first of these solutions decreases and is of the class* $\mathcal{RPV}(-\infty)$ *whereas the second one increases and is of the class* $\mathcal{RPV}(\infty)$*.*

Proof. From Theorem 2.1, (5.20) has solutions $u \in \mathcal{RPV}(-\infty)$ and $v \in \mathcal{RPV}(\infty)$ if and only if (5.21) holds. It is clear from the definition of $\mathcal{RPV}$ functions that $y_1 = u^2 \in \mathcal{RPV}(-\infty)$ and $y_3 = v^2 \in \mathcal{RPV}(\infty)$. □

Note that the third linearly independent solutions $y_2 = uv$ need not to be $\mathcal{RPV}$ at all. This is shown by the example: $u(t) = e^{-t}$, $v(t) = e^{t}$, so that $y_2(t) = 1$.

5.3.2 $\mathcal{RV}$ and $\mathcal{RPV}$ solutions of the perturbed equation

First we establish sufficient conditions for (5.16) to have solutions x_1, x_2, x_3 with the same asymptotic behavior as the solutions $y_1 = u^2$, $y_2 = uv$, $y_3 = v^2$ of (5.17), respectively. Then we apply these results to construct $\mathcal{RV}$ and $\mathcal{RPV}$ solutions of (5.16).

Theorem 5.6. *If for some $\alpha > 0$*

$$\int_a^\infty v^2(s)G(s, \alpha u^2(s))\,\mathrm{d}s < \infty, \tag{5.22}$$

then there exists an eventually positive solution x_1 of (5.16) *such that $x_1(t) \sim \alpha u^2(t)/2$ as $t \to \infty$.*

Proof. Choose $T \geq a$ such that

$$\int_T^\infty v^2(s)G(s, \alpha u^2(s))\,\mathrm{d}s \leq \frac{\alpha}{2},$$

which is possible by (5.22). Define the set $\mathcal{X}_1$ by

$$\mathcal{X}_1 = \{y \in C[T,\infty) : 0 \leq x(t) \leq \alpha u^2(t), t \geq T\}$$

and the integral operator $\mathcal{F}_1$ by

$$\mathcal{F}_1 x(t) = \frac{\alpha}{2}u^2(t) + u^2(t)\int_t^\infty \left(\int_t^s \frac{1}{u^2(s_2)}\int_t^{s_2}\frac{1}{u^2(s_1)}\,\mathrm{d}s_1\,\mathrm{d}s_2\right)u^2(s)F(s,x(s))\,\mathrm{d}s,$$

$t \geq T$. It can be shown that all the hypotheses of the Schauder-Tychonoff fixed point theorem are fulfilled, and thus it ensures the existence of $x_1 \in \mathcal{X}_1$ such that $x_1 = \mathcal{F}_1 x_1$. So x_1 is also a solution of (5.16). The asymptotics of x_1 is an immediate consequence of the integral equation $x_1 = \mathcal{F}_1 x_1$. □

The proofs of the next two theorems are also based on the Schauder-Tychonoff fixed point theorem; we omit details.

Theorem 5.7. *If for some $\beta > 0$*

$$\int_a^\infty u(s)v(s)G(s, \beta u(s)v(s))\,\mathrm{d}s < \infty, \tag{5.23}$$

then there exists an eventually positive solution x_2 of (5.16) *such that $x_2(t) \sim \beta u(t)v(t)/2$ as $t \to \infty$.*

Theorem 5.8. *If for some $\gamma > 0$*

$$\int_a^\infty u^2(s)G\left(s, \gamma v^2(s)\right)\,\mathrm{d}s < \infty, \tag{5.24}$$

then there exists an eventually positive solution x_3 of (5.16) *such that $x_3(t) \sim \gamma v^2(t)/2$ as $t \to \infty$.*

If p satisfies condition (5.18), then the existence of a fundamental set of $\mathcal{RV}$ solutions to (5.17) given by (5.19) is guaranteed. Then, of conditions (5.22), (5.23) and (5.24) hold, Theorems 5.6, 5.7, 5.8, respectively, guarantee the existence of a fundamental set of solutions x_1, x_2, x_3 of (5.16) such that $x_i(t) \sim \alpha_i y_i(t)$ as $t \to \infty$, $\alpha_i > 0$, $i = 1, 2, 3$. But this also means that x_i are in $\mathcal{RV}$. As an illustration, take

$$F(t,x) = t^{\omega} M(t)|x|^{\delta} \operatorname{sgn} x, \ \ \omega \in \mathbb{R}, \ \ \delta \in (0,\infty), \ \ M \in \mathcal{SV},$$

where for a suitable choice of ω, δ the conditions posed on G are fulfilled.

To obtain the existence and asymptotic behavior of $\mathcal{RPV}$ solutions of the perturbed equation (5.16) with $p(t) < 0$, one proceeds in the same way as above in the $\mathcal{RV}$ case.

5.4 Classification of convergence rates of solutions to perturbed first order ODE's

The content of this section is based on selected results from Appleby, Patterson [6]. We are interested in a classification of the rates of convergence to a limit of the solutions of the scalar differential equation

$$x'(t) = -f(x(t)) + g(t), \ \ t > 0, \ \ x(0) = \xi. \tag{5.25}$$

We assume that the unperturbed equation

$$x'(t) = -f(x(t)), \ \ t > 0, \ \ x(0) = \zeta \tag{5.26}$$

has a unique globally stable equilibrium (which we set to be at zero). This is characterized by the condition $xf(x) > 0$ for $x \neq 0$, $f(0) = 0$. Further we assume that $f \in C(\mathbb{R}, \mathbb{R})$, $g \in C([0,\infty), \mathbb{R})$, f is locally Lipschitz continuous on $\mathbb{R}$. We suppose that $f(x)$ does not have linear leading order behavior as $x \to \infty$; moreover, we do not ask that f forces solutions of (5.26) to hit zero in finite time. We define

$$F(x) = \int_x^1 \frac{1}{f(u)} \, \mathrm{d}u, \ \ x > 0,$$

and avoiding solutions of equation (5.26) to hitting zero in finite time forces $\lim_{x \to 0+} F(x) = \infty$. We notice that $F : (0,\infty) \to \mathbb{R}$ is a strictly decreasing function, so it has an inverse F^{-1}, and we have $\lim_{t\to\infty} F^{-1}(t) = 0$. The significance of the functions F and F^{-1} is that they enable us to determine the rate of convergence of solutions of (5.26) to zero, because $F(y(t)) - F(\zeta) = t$ for $t \geq 0$ or $y(t) = F^{-1}(t + F(\zeta))$ for $t \geq 0$. It is then of interest to ask whether solutions of (5.25) will still converge to zero as $t \to \infty$, and how this convergence rate modifies according to the asymptotic behavior of g. In order to do this with reasonable generality we find it convenient and natural to assume at various points that the functions f and g are regularly varying.

The main result of [6], which characterizes the rate of convergence of solutions of (5.25) to zero, can be summarized as follows: Suppose that $f \in \mathcal{RV}_0(\beta)$, $\beta > 1$, and that g is positive and regularly varying at infinity, in such a manner that

$$\lim_{t\to\infty} \frac{g(t)}{(f \circ F^{-1})(t)} =: M \in [0, \infty]$$

exists. If $M = 0$, the solution of (5.25) inherits the rate of decay to zero of y, in the sense that

$$\lim_{t\to\infty} \frac{F(x(t))}{t} = 1.$$

If $M \in (0, \infty)$ we can show that the rate of decay to zero is slightly slower, obeying

$$\lim_{t\to\infty} \frac{F(x(t))}{t} = \Lambda = \Lambda(L) \in (0, 1)$$

and a formula for Λ purely in terms of M and β can be found. Finally, in the case that $M = \infty$ can be shown that

$$\lim_{t\to\infty} \frac{F(x(t))}{t} = 0.$$

If it is presumed that g is regularly varying at infinity with negative index, or g is slowly varying and is asymptotic to a decreasing function, then the exact rate of convergence can be found, namely that $\lim_{t\to\infty} f(x(t))/g(t) = 1$. These asymptotic results are proven by constructing appropriate upper and lower solutions to the differential equation (5.25) as in [3].

In order to simplify the analysis, we assume that $g(t) > 0$, $t > 0$; $x(0) = \xi > 0$. Note that the results can rapidly be extended in the case $g(t) < 0$ and $\xi < 0$.

We will not give the proofs of all the results presented here; note that some of them are rather technical. We prefer to present — for illustration — just a sketch of one selected proof. However, we give several comments.

The first result says that the global convergence of solutions of (5.25), as well as the rate of convergence of solutions to 0 is preserved provided the perturbation g decays sufficiently rapidly. In order to guarantee this, we request only that f be asymptotic to a monotone function close to zero.

Theorem 5.9 (Appleby, Patterson [6]). *Let there exist ϕ such that*

$$\lim_{t\to 0+} \frac{f(t)}{\phi(t)} = 1, \ \ \phi \textit{ is increasing on } (0, \delta). \tag{5.27}$$

If

$$\lim_{t\to\infty} \frac{g(t)}{(f \circ F^{-1})(t)} = 0, \tag{5.28}$$

then the unique continuous solution of (5.25) *obeys*

$$\lim_{t\to\infty} \frac{F(x(t))}{t} = 1. \tag{5.29}$$

Immediately this theorem presents a question: Is it possible to find slower rates of decay of $g(t) \to 0$ as $t \to \infty$ than exhibited in (5.28), for which the solution x of (5.25) still decays at the rate of the unperturbed equation, as characterized by (5.29)? In some sense, the next theorem says that the rate of decay of g in (5.28) cannot be relaxed, at least for functions f which are regularly varying at zero with index $\beta > 1$, or which are rapidly varying at zero.

In the case when f is regularly varying at 0 with index 1 (and $f(x)/x \to 0$ as $x \to 0$), the condition (5.28) is not necessary in order to preserve the rate of decay embodied by (5.29). A more careful analysis is needed to characterize the asymptotic behavior of solutions of (5.25).

Theorem 5.10 (Appleby, Patterson [6]). *Let x be the unique continuous solution of* (5.25). *Suppose that there exists ϕ such that* (2.24) *holds, and suppose further that there exists $M > 0$ such that*

$$\lim_{t\to\infty} \frac{g(t)}{(f \circ F^{-1})(t)} = M. \tag{5.30}$$

Then $x(t) \to 0$ as $t \to \infty$.

(i) If $f \in \mathcal{RV}(\beta)$ for $\beta > 1$, then

$$\lim_{t\to\infty} \frac{F(x(t))}{t} = \Lambda_*(M) \in (0, 1), \tag{5.31}$$

where Λ_* is the unique solution of $(1 - \Lambda_*)\Lambda_*^{-\beta/(\beta-1)} = M$.

(ii) If $f \circ F^{-1} \in \mathcal{RV}(-1)$, then

$$\lim_{t\to\infty} \frac{F(x(t))}{t} = \Lambda_*(M) \in (0, 1),$$

where Λ_* is the unique solution of $(1 - \Lambda_*)/\Lambda_* = M$.

If y is the solution of (5.26), we have $y(t)/F^{-1}(t) \to 1$ as $t \to \infty$. Moreover, in the case when $\beta > 1$, as $F^{-1} \in \mathcal{RV}(-1/(\beta - 1))$, we have

$$\lim_{t\to\infty} \frac{x(t)}{y(t)} = \lim_{t\to\infty} \frac{x(t)}{F^{-1}(t)} \lim_{t\to\infty} \frac{F^{-1}(\Lambda_* t)}{F^{-1}(t)} = \Lambda_*^{-\beta/(\beta-1)} > 1.$$

Therefore, the solution of (5.25) is of the same order as the solution of (5.26), but decays more slowly by a factor depending on M. In the second case, when $F^{-1} \in \mathcal{SV}$, we have

$$\lim_{t\to\infty} \frac{x(t)}{y(t)} = \lim_{t\to\infty} \frac{x(t)}{F^{-1}(t)} \lim_{t\to\infty} \frac{F^{-1}(\Lambda_* t)}{F^{-1}(t)} = 1,$$

so once again the solution of (5.25) is of the same order as the solution of (5.26)

There is a greater alignment of the hypotheses that appears at a first glance. When $f \in \mathcal{RV}_0(\beta)$ for $\beta > 1$, it follows that $F \in \mathcal{RV}_0(1 - \beta)$ and therefore that $F^{-1} \in \mathcal{RV}(-1/(\beta-1))$ and $f \circ F^{-1} \in \mathcal{RV}(-\beta/(\beta-1))$. Hence we see that the hypothesis of part (ii) are in some sense the limit of those in part (i) when $\beta \to \infty$. This suggests that part (ii) of the theorem applies in the case when f is a rapidly varying function at 0, and the solutions of the unperturbed differential equation are slowly varying at infinity. Moreover, the solution of the perturbed differential equation should also be slowly varying in this case. If we suppose that $f \circ F^{-1} \in \mathcal{RV}(-1)$, then $F^{-1} \in \mathcal{SV}$. Therefore, we do not need to assume this second hypothesis in part (ii) of the above theorem. Further note that if $f \in \mathcal{RPV}_0(\infty)$, then $F^{-1} \in \mathcal{SV}$, see [6].

We notice that viewed as a function of M, $\Lambda_* : (0,\infty) \to (0,1)$ is decreasing and continuous with $\lim_{M\to 0+} \Lambda_*(M) = 1$, $\lim_{M\to\infty} \Lambda_*(M) = 0$. The first limit demonstrates that the limit in (5.31) is a continuous extension of the limit observed in Theorem 5.9, because the hypothesis (5.28) can be viewed as (5.30) with $M = 0$, while the resulting limiting behavior of the solution (5.29) can be viewed as (5.31) where $\Lambda_* = 1$. The monotonicity of Λ_* in M indicates that the slower the decay rate of the perturbation is (i.e., the greater is M) the slower the rate of decay of the solution of (5.25) is. Since $\lim_{M\to\infty} \Lambda_*(M) = 0$, this result also suggests that

$$\lim_{t\to\infty} \frac{g(t)}{(f \circ F^{-1})(t)} = \infty \tag{5.32}$$

implies

$$\lim_{t\to\infty} \frac{F(x(t))}{t} = 0, \tag{5.33}$$

so that the solution of the perturbed differential equation entirely loses the decay properties of the underlying unperturbed equation when the perturbation g exceeds the critical size indicated by (5.30), and decays more slowly yet. This conjecture is borne out by virtue of the next theorem.

Theorem 5.11. *Let x be the unique continuous solution of* (5.25). *Suppose that there exists ϕ such that* (2.24) *holds, and suppose further that f and g obey* (5.32). *Suppose finally that $x(t) \to 0$ as $\to \infty$. If $f \in \mathcal{RV}_0(\beta)$ for $\beta > 1$ or $f \circ F^{-1} \in \mathcal{RV}(-1)$, then the unique solution of* (5.25) *obeys* (5.33).

Proof. We give only a sketch of the proof when $f \circ F^{-1} \in \mathcal{RV}(-1)$ and $F^{-1} \in \mathcal{SV}$. The other part is proved similarly. For details see [6]. Since $f(x)/\phi(x) \to 1$ as $x \to 0+$ and $F^{-1}(t) \to 0$ as $t \to \infty$, we have

$$\lim_{t\to\infty} \frac{\phi(F^{-1}(t))}{f(F^{-1}(t))} = 1.$$

Hence $h = \phi \circ F^{-1} \in \mathcal{RV}(-1)$. Let $\varepsilon \in (0, 1/2)$. By (5.32), we have that there exists $T_1(\varepsilon) > 0$ such that $h(t) < \varepsilon^2 g(t)$ for $t \geq T_1(\varepsilon)$. Also, as $h \in \mathcal{RV}(-1)$, we have that $h(\varepsilon t)/h(t) \to 1/\varepsilon$ as $t \to \infty$. Hence there exists $T_2(\varepsilon) > 0$ such that $h(\varepsilon t) < 2h(t)/\varepsilon$ for $t \geq T_2(\varepsilon)$. Define $T(\varepsilon) = 1 + \max\{T_1(\varepsilon), T_2(\varepsilon)\}$. Now define

$$K = \max\left\{2, \frac{2x(T)}{x_1(\varepsilon)}, \frac{x(T)}{F^{-1}(\varepsilon T)}\right\},$$

where $x_1(\varepsilon) > 0$ is such that $f(x) < (1+\varepsilon)\phi(x)$ for $x \le x_1(\varepsilon)$, and also define

$$x_L(t) = F^{-1}[\varepsilon(t-T) + F(x(T)/K)],$$

$t \ge T$. It can be shown that $x_L'(t) < -f(x_L(t)) + g(t)$, $t \ge T$; $x_L(T) < x(T)$. Therefore, we have that $x_L(t) < x(t)$ for $t \ge T$. One can then get

$$0 \le \liminf_{t\to\infty} \frac{F(x(t))}{t} \le \limsup_{t\to\infty} \frac{F(x(t))}{t} \le \varepsilon.$$

Letting $\varepsilon \to 0+$ gives (5.33). □

We observe that the hypothesis that $x(t) \to 0$ as $\to \infty$ has been appended to the theorem. This is because the slow rate of decay of g may now cause solutions to tend to infinity, if coupled with a hypothesis on f which forces $f(x)$ to tend to zero as $x \to \infty$ at a sufficiently rapid rate. We prefer to add this hypothesis, rather than sufficient conditions on f and g which would guarantee $x(t) \to \infty$.

Several further situations are considered in [6]. For instance, g is assumed to be regularly varying at infinity.

5.5 Nearly linear differential equations

The results of this section are taken from the paper [149] by Řehák. We consider the nonlinear equation

$$(G(y'))' = p(t)F(y), \tag{5.34}$$

where p is a positive continuous function on $[a,\infty)$ and F, G are continuous functions on $\mathbb{R}$ with $uF(u) > 0, uG(u) > 0$ for $u \ne 0$. To simplify our considerations we suppose that F and G are increasing and odd. Nonlinearities F and G are further assumed to have regularly varying behavior of index 1 at zero. More precisely, we require

$$F(|\cdot|), G(|\cdot|) \in \mathcal{RV}_0(1); \tag{5.35}$$

the class $\mathcal{RV}_0$ being defined below. This condition justifies the terminology a *nearly linear equation*. Indeed, if we make a trivial choice $F = G = \mathrm{id}$, then (5.34) reduces to a linear equation. It is however clear that in contrast to linear equations, the solution space of (2.1) is generally neither additive nor homogeneous. Examples of $F(u)$ and $G(u)$ which lead to a nonlinear equation and can be treated within this theory are: $u|\ln|u||$, or $u/|\ln|u||$, or $u/\sqrt{1 \pm u^2}$, and many others.

As we could see in Chapter 4, theory of regular variation has been shown very useful in studying asymptotic properties of Emden-Fowler type equations, e.g. of the form $y'' = q(t)|y|^\gamma \operatorname{sgn} y$ or, more generally,

$$y'' = q(t)\varphi(y), \tag{5.36}$$

where $|\varphi(|\cdot|)| \in \mathcal{RV}(\gamma)$ or $|\varphi(|\cdot|)| \in \mathcal{RV}_0(\gamma)$, $\gamma > 0$. Usually the sub-linearity condition resp. the super-linearity condition is assumed, i.e., $\gamma < 1$ resp. $\gamma > 1$, and such conditions play an important role in the proofs. Notice that from this point of view, equation (5.34) (which arises as a variant of (5.36) with specific nonlinearities) is neither super-linear nor sub-linear, since the indices of regular variation of F and G are the same. Therefore, asymptotic analysis of (5.34) in the framework of regular variation requires an approach which is different from the usual ones for the above mentioned Emden-Fowler equation with $\gamma \neq 1$. The crucial property is now the fact that the nonlinearities in (5.34) are somehow close to each other (they can differ by a slowly varying function). It turns out that a modification of some methods known from the linear theory is a useful tool. However, as we will see, some phenomena may occur for (5.34) which cannot happen in the linear case.

We are interested in asymptotic behavior of solutions y to (5.34) such that $y(t)y'(t) < 0$ for large t. Without loss of generality, we restrict our study to eventually positive decreasing solutions of equation (5.34); such a set is denoted as $\mathcal{DS}$. As we will see, for any $y \in \mathcal{DS}$, $\lim_{t\to\infty} y'(t) = 0$.

We start with the simple result which gives the conditions guaranteeing slow variation of any solution in $\mathcal{DS}$.

Theorem 5.12. *Assume that*

$$\lim_{t\to\infty} t \int_t^\infty p(s)\,\mathrm{d}s = 0, \tag{5.37}$$

$$\limsup_{u\to 0+} L_F(u) < \infty \quad \textit{and} \quad \liminf_{u\to 0+} L_G(u) > 0. \tag{5.38}$$

Then

$$\emptyset \neq \mathcal{DS} \subset \mathcal{NSV}.$$

Proof. Rewrite equation (5.34) as an equivalent system of the form

$$y' = -G^{-1}(u), \quad u' = -p(t)F(y),$$

where G^{-1} is the inverse of G. Then we apply the existence result [20, Theorem 1] to obtain $\mathcal{DS} \neq \emptyset$.

Take $y \in \mathcal{DS}$, i.e., $y(t) > 0$, $y'(t) < 0$, $t \geq t_0$. Then $\lim_{t\to\infty} y'(t) = 0$. Indeed, $G(y')$ is negative increasing and so is y'. If $\lim_{t\to\infty} y'(t) = -c < 0$, then $y(t) - y(t_0) \sim -c(t - t_0)$ as $t \to \infty$, which contradicts eventual positivity of y. Integration of (5.34) from t to ∞ yields

$$-G(y'(t)) = \int_t^\infty p(s)F(y(s))\,\mathrm{d}s.$$

Hence,

$$|y'(t)|L_G(|y'(t)|) = \int_t^\infty p(s)y(s)L_F(y(s))\,\mathrm{d}s \leq y(t) \int_t^\infty p(s)L_F(y(s))\,\mathrm{d}s.$$

Thus,

$$\frac{-ty'(t)}{y(t)} \le \frac{t}{L_G(|y'(t)|)} \int_t^\infty p(s)L_F(y(s))\,\mathrm{d}s \le \frac{tM}{N} \int_t^\infty p(s)\mathrm{d}s, \tag{5.39}$$

where M, N are some positive constants which exist thanks to (5.38). Since the expression on the right hand side of (5.39) tends to zero, it follows that $ty'(t)/y(t) \to 0$ as $t \to \infty$, and $y \in \mathcal{NSV}$ follows. $\square$

Remark 5.3. Condition (5.37) is somehow necessary. Indeed, take $y \in \mathcal{DS} \cap \mathcal{SV}$ and assume that $\liminf_{u\to 0+} L_F(u) > 0$ and $\limsup_{u\to 0+} L_G(u) < \infty$. First note that because of monotonicity of y', we have $ty'(t)/y(t) \to 0$, and so $y \in \mathcal{NSV}$. Set $w = G(y')/y$. Then w satisfies

$$w' = p(t)\frac{F(y)}{y} - w\frac{y'}{y} \tag{5.40}$$

for large t. There exists $N \in (0, \infty)$ such that

$$0 < -tw(t) \le -Nt\frac{y'(t)}{y(t)} \to 0$$

as $t \to \infty$. Hence,

$$\int^\infty w(s)\frac{y'(s)}{y(s)}\,\mathrm{d}s < \infty \quad \text{and} \quad \lim_{t\to\infty} t \int_t^\infty w(s)\frac{y'(s)}{y(s)}\,\mathrm{d}s = 0.$$

Integration (5.40) from t to ∞ and multiplying by t, we get

$$-tw(t) = t\int_t^\infty p(s)\frac{F(y(s))}{y(s)}\,\mathrm{d}s - t\int_t^\infty w(s)\frac{y'(s)}{y(s)}\,\mathrm{d}s,$$

which implies $\lim_{t\to\infty} t\int_t^\infty p(s)L_F(y(s))\,\mathrm{d}s = 0$. Since $M \in (0, \infty)$ exists such that $L_F(y(t)) \ge M$ for large t, condition (5.37) follows.

A necessity is discussed from certain point of view also in Remark 5.6.

Remark 5.4. Observe that in Theorem 5.12 we are dealing with all $\mathcal{SV}$ solutions of (5.34). It follows from the fact that $\mathcal{SV}$ solutions cannot increase. Indeed, for a positive increasing solution u of (5.34), due to convexity, we have $u'(t) \ge K_1$ for some $K_1 > 0$. By integrating, $u(t) \ge K_1 t + K_2$, which contradicts the fact the $u \in \mathcal{SV}$.

Remark 5.5. The statements of Theorem 5.12 and Remark 5.3 can be understood as a nonlinear extension of Theorem 2.1-(i).

In the next result, we derive asymptotic formulae for $\mathcal{SV}$ solutions provided p is regularly varying of index -2. Define

$$\hat{F}(x) = \int_1^x \frac{\mathrm{d}u}{F(u)}, \quad x > 0.$$

The function $\hat{F}(x)$ is increasing on $(0,\infty)$. The constant 1 in the integral is unimportant; it can be replaced by any positive constant. Denote the inverse of $\hat{F}$ by $\hat{F}^{-1}$. We have $|\hat{F}| \in \mathcal{SV}_0$ and in general $\lim_{u\to 0+} |\hat{F}(u)|$ can be finite or infinite. Denote

$$H(t) = \frac{tp(t)}{L_G(1/t)}$$

and note that $H \in \mathcal{RV}(-1)$ provided $p \in \mathcal{RV}(-2)$.

Theorem 5.13. *Assume that* $p \in \mathcal{RV}(-2)$, $\lim_{u\to 0+} |\hat{F}(u)| = \infty$, *and*

$$L_G(ug(u)) \sim L_G(u) \quad \text{as } u \to 0+, \tag{5.41}$$

for all $g \in \mathcal{SV}_0$. *If* $y \in \mathcal{DS} \cap \mathcal{SV}$, *then* $-y \in \Pi(-ty'(t))$. *Moreover:*
(i) If $\int_a^\infty H(s)\,\mathrm{d}s = \infty$, *then*

$$y(t) = \hat{F}^{-1}\left\{-\int_a^t (1+o(1))H(s)\,\mathrm{d}s\right\} \tag{5.42}$$

(and $y(t) \to 0$*) as* $t \to \infty$.
(ii) If $\int_a^\infty H(s)\,\mathrm{d}s < \infty$, *then*

$$y(t) = \hat{F}^{-1}\left\{\hat{F}(y(\infty)) + \int_t^\infty (1+o(1))H(s)\,\mathrm{d}s\right\} \tag{5.43}$$

(and $y(t) \to y(\infty) \in (0,\infty)$*) as* $t \to \infty$.

Proof. Take $y \in \mathcal{DS} \cap \mathcal{SV}$ and let t_0 be such that $y(t) > 0, y'(t) < 0$ for $t \ge t_0$. Then

$$(G(y'))' = pF(y) \in \mathcal{RV}(-2 + 1\cdot 0) = \mathcal{RV}(-2)$$

provided $y(t) \to 0$ as $t \to \infty$. If $y(t) \to C \in (0,\infty)$, then we get the same conclusion since $F(y(t)) \to F(C) \in (0,\infty)$, and so $pF(y) \in \mathcal{RV}(-2)$. Thus

$$G(-y'(t)) = -G(y'(t)) = \int_t^\infty (G(y'(s)))'\mathrm{d}s \in \mathcal{RV}(-1).$$

In view of $-y' = G^{-1}(G(-y'))$, we get $-y' \in \mathcal{RV}(-1)$. Hence,

$$\frac{-y(\lambda t) + y(t)}{-ty'(t)} = \int_t^{\lambda t} \frac{-y'(u)}{-ty'(t)}\,\mathrm{d}u = \int_1^\lambda \frac{-y'(st)}{-y'(t)}\,\mathrm{d}s \to \int_1^\lambda \frac{\mathrm{d}s}{s} = \ln\lambda \tag{5.44}$$

as $t \to \infty$, thanks to the uniformity. This implies $-y \in \Pi(-ty'(t))$. Define

$$\Psi(t) = tG(y'(t)) - \int_{t_0}^t G(y'(s))\,\mathrm{d}s.$$

Then $\Psi'(t) = tp(t)F(y(t)) \in \mathcal{RV}(-1)$, which implies $\Psi \in \Pi(t\Psi'(t))$, similarly as in (5.44). Further, we claim $\Psi \in \Pi(-tG(y'(t)))$. Indeed, fix $\lambda > 0$, and then

$$\begin{aligned}\frac{\Psi(\lambda t) - \Psi(t)}{-tG(y'(t))} &= \frac{\lambda G(y'(\lambda t))}{-G(y'(t))} + 1 - \frac{\int_t^{\lambda t} G(y'(s))\,\mathrm{d}s}{-tG(y'(t))} \\ &= \frac{\lambda G(y'(\lambda t))}{-G(y'(t))} + 1 + \int_1^{\lambda} \frac{G(y'(tu))}{G(y'(t))}\,\mathrm{d}u \to \int_1^{\lambda} \frac{\mathrm{d}u}{u} = \ln \lambda\end{aligned}$$

as $t \to \infty$, thanks to $G(-y') \in \mathcal{RV}(-1)$ and the uniformity. From the uniqueness of the auxiliary function up to asymptotic equivalence, we obtain

$$-G(y'(t)) \sim tp(t)F(y(t)) \tag{5.45}$$

as $t \to \infty$. Condition (5.41) is equivalent to $L_G(v(t)/t) \sim L_G(1/t)$ as $t \to \infty$, for all $v \in \mathcal{SV}$. Hence,

$$-G(y'(t)) = -y'(t)L_G(L_{|y'|}(t)/t) \sim -y'(t)L_G(1/t)$$

as $t \to \infty$. Combining this relation with (5.45), we get

$$\frac{-y'(t)}{F(y(t))} \sim \frac{tp(t)}{L_G(1/t)}$$

as $t \to \infty$, that is

$$\frac{y'(t)}{F(y(t))} = -(1 + o(1))H(t) \tag{5.46}$$

as $t \to \infty$. By integrating this relation over (t_0, t) we obtain

$$\hat{F}(y(t)) = \hat{F}(y(t_0)) - \int_{t_0}^{t} (1 + o(1))H(s)\,\mathrm{d}s, \tag{5.47}$$

which implies (5.42) provided $\int_a^\infty H(s)\,\mathrm{d}s = \infty$. Clearly then $y(t) \to 0$ as $t \to \infty$, otherwise we get a contradiction with the divergence of the integral in (5.47). If $\int_a^\infty H(s)\,\mathrm{d}s < \infty$ holds, then we integrate (5.46) over (t, ∞) obtaining (5.43). In this case, $y(t)$ must tend to a positive constant as $t \to \infty$. Indeed, if $y(t) \to 0$ as $t \to \infty$, then the left-hand side of (5.47) becomes unbounded which is a contradiction. □

Remark 5.6. A closer examination of the proof of Theorem 5.13 shows that the condition $\lim_{u\to 0+} |\hat{F}(u)| = \infty$ is somehow needed. Indeed, if we assume that this limit is finite and that $\int_a^\infty H(s)\,\mathrm{d}s = \infty$, then in view of (5.47) we get contradiction. As a by-product we then have a non-existence of $\mathcal{SV}$ solutions. If $\lim_{u\to 0+} |\hat{F}(u)| < \infty$ holds when $\int_a^\infty H(s)\,\mathrm{d}s < \infty$, then no conclusion whether $y(\infty) = 0$ or $y(\infty) > 0$ can generally be drawn. Note that such phenomena cannot occur in the linear case.

Remark 5.7. There exists an alternative way how to prove (5.45). Indeed, denote $\tilde{L}(t) = L_p(t)F(y(t))$ and observe that $\tilde{L} \in \mathcal{SV}$. Therefore,

$$\int_t^\infty p(s)F(y'(s))\,\mathrm{d}s = \int_t^\infty s^{-2}\tilde{L}(s)\,\mathrm{d}s \sim \frac{1}{t}\tilde{L}(t) = tp(t)F(y(t))$$

as $t \to \infty$ by the Karamata theorem. Since $-G(y'(t)) = \int_t^\infty p(s)F(y(s))\,\mathrm{d}s$, we obtain (5.45).

Remark 5.8. Observe that to prove asymptotic formulae for decreasing $\mathcal{SV}$ solutions of (5.34) we do not require (even one-sided) boundedness conditions on L_F and L_G such as (5.38). As for condition (5.41) from Theorem 5.13, it is not too restrictive. Many functions satisfy it, for example, $L_G(u) \to C \in (0, \infty)$ as $u \to 0+$, or $L_G(u) = |\ln|u||^{\alpha_1}|\ln|\ln|u|||^{\alpha_2}$, $\alpha_1, \alpha_2 \in \mathbb{R}$. Compare also with (4.23).

The value -2 in the condition $p \in \mathcal{RV}(-2)$ is natural and consistent within our setting. Indeed, since we work with $\mathcal{SV}$ solutions, the expression on the left-hand side of (5.34), which is somehow close to the second derivative, is expected to be in $\mathcal{RV}(-2)$.

Corollary 5.1. *Assume that $p \in \mathcal{RV}(-2)$ and $\lim_{t\to\infty} L_p(t) = 0$. Let* (5.38) *and* (5.41) *hold. Then any solution $y \in \mathcal{DS}$ belongs to $\mathcal{NSV}$. Moreover, $-y \in \Pi(-ty'(t))$ and asymptotic formulae* (5.42) *or* (5.43) *hold.*

Proof. By the Karamata theorem,

$$t\int_t^\infty p(s)\,\mathrm{d}s = t\int_t^\infty s^{-2}L_p(s)\,\mathrm{d}s \sim L_p(t) \to 0$$

as $t \to \infty$, and so (5.37) follows. Further, in view of $\limsup_{u\to 0+} L_F(u) < \infty$, exists $M > 0$ such that we have for $x < 1$,

$$\hat{F}(x) \le -\int_x^1 \frac{\mathrm{d}u}{uM} = \frac{\ln x}{M},$$

which implies $\lim_{x\to 0+} \hat{F}(x) = -\infty$. The statement now follows from Theorem 5.12 and Theorem 5.13. □

Remark 5.9. Corollary 5.1 can be seen as a nonlinear extension of Theorem 2.11.

Example 5.1. Consider the equation

$$(y'L_G(|y'|))' = \frac{L_p(t)y}{t^2|\ln|y||}, \tag{5.48}$$

where $L_G \in \mathcal{SV}_0$ and $L_p \in \mathcal{SV}$. Then $\hat{F}(x) = \frac{-(\ln x)^2}{2}$, $x \in (0, 1)$, $\hat{F}(x) \to -\infty$ as $x \to 0+$, and $\hat{F}^{-1}(u) = \exp\{-\sqrt{-2u}\}$, $u < 0$. We restrict our considerations to positive (decreasing) solutions y of (5.48) such that $y(t) < 1$ for $t \ge t_0$; we have this requirement because we need $F(u)$ to be increasing at least in a certain neighborhood of zero (here it is $(0,1)$). Note that a slight modification of F, namely $F(u) = x/|\ln|x/k||$, $k \in (0,\infty)$, ensures the required monotonicity of F on the (possibly bigger) interval $(0,k)$.

(i) Let $G(u) = u|\ln|u||$ and $L_p(t) = \frac{1}{\ln t + h(t)}$, where h is a continuous function on $[a,\infty)$ with $|h(t)| = o(\ln t)$ as $t \to \infty$, and such that $\ln t + h(t) > 0$ for $t \in [a,\infty)$. Examples of h are $h(t) = \cos t$ or $h(t) = \ln(\ln t)$. Note that the required monotonicity of G is ensured in a small neighborhood of zero But this is not a problem since we have y' as the argument of G, and $y'(t)$ tends to zero as $t \to \infty$. Nevertheless, we could modify G similarly as in the above mentioned modification of F. The function H reads as

$$H(t) = \frac{1}{t(\ln t + h(t))|\ln(1/t)|} = \frac{1}{t(\ln t + h(t))\ln t} \sim \frac{1}{t(\ln t)^2}$$

as $t \to \infty$. Thus, $\int^\infty H(s)\,ds < \infty$ and we have $\int_t^\infty H(s)\,ds \sim \frac{1}{\ln t}$ as $t \to \infty$. From Corollary 5.1, we get that for any eventually decreasing positive solution y of (5.48) (with $y(t) < 1$ for large t), $-y$ is in Π (y is in $\mathcal{NSV}$), y tends to $y(\infty) > 0$ and satisfies the formula

$$y(t) = \exp\left\{-\sqrt{(\ln y(\infty))^2 - \frac{2(1+o(1))}{\ln t}}\right\}$$

as $t \to \infty$.

(ii) Let L_p be the same as in (i) and $G(u) = \frac{u}{\sqrt{1 \pm u^2}}$. Then

$$H(t) = \frac{\sqrt{1 \pm \frac{1}{t^2}}}{t(\ln t + h(t))} \sim \frac{1}{t\ln t}$$

as $t \to \infty$. Note that $(\ln(\ln t))' = \frac{1}{t\ln t}$, and so $\int^\infty H(s)\,ds = \infty$. From Corollary 5.1, we get that for any eventually decreasing positive solution y of (5.48), $-y$ is in Π (y is in $\mathcal{NSV}$), y tends to zero and satisfies the formula

$$y(t) = \exp\left\{-\sqrt{(1+o(1))\ln(\ln t)}\right\}$$

as $t \to \infty$.

(iii) Let $L_p(t) = \frac{1}{(\ln t + h(t))^2}$, where h is as in (i), and $G = \mathrm{id}$. Then

$$H(t) = \frac{1}{t(\ln t + h(t))^2} \sim \frac{1}{t(\ln t)^2}$$

as $t \to \infty$. Applying Corollary 5.1, we get that any eventually decreasing positive solution y of (5.48) (with $y(t) < 1$ for large t) obeys the same asymptotic behavior as y in (i).

Among others, the above examples show how the convergence / divergence of the integral $\int^\infty H(s)\,ds$ can be affected by the behavior of both p and G.

Under the conditions of Theorem 5.13-(i), it does not follow that

$$y(t) \sim \hat{F}^{-1}\left\{-\int_a^t H(s)\,\mathrm{d}s\right\}$$

as $t \to \infty$; this fact was observed already in the linear case, see [45, Remark 2], see also the text after Theorem 2.11. However, we can give a lower estimate under quite mild assumptions. For technical reasons we consider positive decreasing solutions (5.34) on $[0,\infty)$ (provided $p \in C([0,\infty))$).

Theorem 5.14. *(i) Let* $\liminf_{u\to 0+} L_G(u) > 0$ *and* (5.37) *hold. Then* $y \in \mathcal{DS} \cap \mathcal{SV}$ *satisfies the estimate*

$$\liminf_{t\to\infty} \frac{y(t)}{\hat{F}^{-1}\left\{\hat{F}(y(0)) - M\int_0^t sp(s)\,\mathrm{d}s\right\}} \geq 1, \tag{5.49}$$

where M is some positive constant. The constant M can be taken such that $M = 1/\inf_{u\in[0,|y'(0)|]} L_G(u)$.

(ii) In addition to the conditions in (i), assume that $\limsup_{u\to 0+} L_F(u) < \infty$ *holds. Then* $y \in \mathcal{DS}$ *implies* $y \in \mathcal{NSV}$ *and*

$$\liminf_{t\to\infty} y(t)\exp\left\{N\int_0^t sp(s)\,\mathrm{d}s\right\} \geq y(0),$$

where N is some positive constant. The constant N can be taken such that $N = \sup_{u\in[0,y(0)]} L_F(u)/\inf_{u\in[0,|y'(0)|]} L_G(u)$.

Proof. (i) Take $y(t) \in \mathcal{DS} \cap \mathcal{SV}$, $t \geq 0$. For $\lambda \in (0,1)$, we have

$$\frac{-G(y'(\lambda t)) + G(y'(t))}{F(y(\lambda t))} = \frac{1}{F(y(\lambda t))}\int_{\lambda t}^t p(s)F(y(s))\,\mathrm{d}s \leq \int_{\lambda t}^t p(s)\,\mathrm{d}s, \tag{5.50}$$

$t > 0$. Thanks to $\liminf_{u\to 0+} L_G(u) > 0$, there exists $M > 0$ such that

$$\frac{-y'(\lambda t)}{MF(y(\lambda t))} + \frac{G(y'(t))}{F(y(\lambda t))} \leq \frac{-y'(\lambda t)L_G(|y'(t)|)}{F(y(\lambda t))} + \frac{G(y'(t))}{F(y(\lambda t))} \leq \int_{\lambda t}^t p(s)\,\mathrm{d}s, \tag{5.51}$$

$t > 0$, where the last estimate follows from (5.50). Integration over $\lambda \in (0,1)$ yields

$$\frac{-1}{Mt}[\hat{F}(y(t)) - \hat{F}(y(0))] + \frac{G(y'(t))}{t}\int_0^t \frac{\mathrm{d}s}{F(y(s))} \leq \frac{1}{t}\int_0^t sp(s)\,\mathrm{d}s, \tag{5.52}$$

where we substituted $s = \lambda t$ in $\int_0^1 \frac{\mathrm{d}\lambda}{F(y(\lambda t))}$ and we applied the Fubini theorem in $\int_0^1\int_{\lambda t}^t p(s)\,\mathrm{d}s\,\mathrm{d}\lambda$. From (5.52), we get

$$y(t) \geq \hat{F}^{-1}\left\{\hat{F}(y(0)) + MG(y'(t))\int_0^t \frac{\mathrm{d}s}{F(y(s))} - M\int_0^t sp(s)\,\mathrm{d}s\right\}. \tag{5.53}$$

Since $F(y) \in \mathcal{SV}$, the Karamata theorem yields

$$0 < -G(y'(t)) \int_0^t \frac{\mathrm{d}s}{F(y(s))} \sim \frac{-tG(y'(t))}{F(y(t))}$$
$$= \frac{t}{F(y(t))} \int_t^\infty p(s)F(y(s))\,\mathrm{d}s \le t \int_t^\infty p(s)\,\mathrm{d}s,$$

where the asymptotic relation holds as $t \to \infty$. Hence,

$$-G(y'(t)) \int_0^t \frac{\mathrm{d}s}{F(y(s))} = o(1)$$

as $t \to \infty$. Formula (5.49) now easily follows from (5.53).

(ii) Take $y(t) \in \mathcal{DS}$, $t > 0$. Then $y \in \mathcal{NSV}$ follows from Theorem 5.12. Thanks to (5.38) which is in fact assumed, there exists $N > 0$ such that $-NG(y'(\lambda t))/F(y(\lambda t)) \ge -y'(\lambda t)/y(\lambda t)$, $t > 0$. As in the proof of (i), we then get

$$\frac{-y'(\lambda t)}{y(\lambda t)} + \frac{G(y'(t))}{F(y(\lambda t))} \le \int_{\lambda t}^t p(s)\,\mathrm{d}s.$$

Since this estimate is a special case of (5.51), the rest of the proof is now clear. □

Remark 5.10. It is reasonable to require the conditions $\lim_{u\to 0+} |\hat{F}(u)| = \infty$ and $\int_0^\infty sp(s)\,\mathrm{d}s = \infty$ when applying Theorem 5.14. Further notice that the proof of Theorem 5.14 does not require $p \in \mathcal{RV}(-2)$, in contrast to the approach known from the linear case, cf. [45, Remark 2]. From this point of view, the result is an improvement even in the linear case. Nevertheless, in order to see Theorem 5.14 as a partial refinement of information about solutions treated in Theorem 5.13-(i), it is reasonable to assume $p \in \mathcal{RV}(-2)$.

We now consider more general equation

$$(r(t)G(y'))' = p(t)F(y), \tag{5.54}$$

where r and p are positive continuous functions on $[a, \infty)$ and F, G are as before. First note that in the case when $G = \mathrm{id}$ and $\int_a^\infty 1/r(s)\,\mathrm{d}s = \infty$, equation (5.54) can be transformed into the equation of the form (2.1) and the type of the interval (on which the equation is considered) is preserved. Indeed, denote $R(t) = \int_a^t 1/r(s)\,\mathrm{d}s$ and introduce new independent variable $s = R(t)$ and new function $z(s) = y(R^{-1}(t))$. Then (5.54) is transformed into

$$\frac{\mathrm{d}^2 z}{\mathrm{d}s^2} = \tilde{p}(s)F(z), \quad \text{where} \quad \tilde{p}(s) = p(R^{-1}(s))r(R^{-1}(s)),$$

$s \in [R(a), \infty)$. For a general G however such a transformation is not at disposal, and we must proceed directly. Let $\mathcal{DS}_r$ denote the set of all eventually positive decreasing solutions of equation (5.54). An extension of Theorem 5.12 to (5.54) reads as follows.

Theorem 5.15. *Assume that*

$$\lim_{t\to\infty} \frac{t}{r(t)} \int_t^\infty p(s)\,\mathrm{d}s = 0, \tag{5.55}$$

$\limsup_{u\to 0+} L_F(u) < \infty$,

$$\int_a^\infty G^{-1}\left(\frac{M}{r(s)}\right) \mathrm{d}s = \infty \tag{5.56}$$

for all $M \in (0,\infty)$, *and* $L_G(u) \ge N$, $u \in (0,\infty)$, *for some* $N > 0$. *Then* $\emptyset \ne \mathcal{DS}_r \subset \mathcal{NSV}$.

Proof. We give only a concise proof. Existence of solutions in $\mathcal{DS}_r$ again follows from [20]. Take $y \in \mathcal{DS}_r$. Then $r(t)G(y'(t)) \to 0$ as $t \to \infty$. Otherwise we get contradiction with eventual positivity of y, because of condition (5.56). Similarly as in the proof of Theorem 5.12 we get that there exists $K \in (0,\infty)$ such that

$$\frac{-ty'(t)}{y(t)} \le \frac{tL_F(y(t))}{L_G(|y'(t)|)r(t)} \int_t^\infty p(s)\,\mathrm{d}s \le \frac{tK}{r(t)} \int_t^\infty p(s)\,\mathrm{d}s$$

for large t. Hence, $y \in \mathcal{NSV}$. □

For $p \in \mathcal{RV}(\beta)$ and $r \in \mathcal{RV}(\beta+2)$ with $\beta < -1$, denote

$$H_r(t) = \frac{tp(t)}{(-\beta-1)r(t)}$$

and note that then $H_r \in \mathcal{RV}(-1)$. An extension of Theorem 5.13 to (5.54) reads as follows.

Theorem 5.16. *Assume that* $p \in \mathcal{RV}(\beta)$ *and* $r \in \mathcal{RV}(\beta+2)$, *with* $\beta < -1$. *Further, let* $\lim_{u\to 0+} |\hat{F}(u)| = \infty$ *and* (5.41) *hold. If* $y \in \mathcal{DS}_r \cap \mathcal{SV}$, *then* $-y \in \Pi(-ty'(t))$. *Moreover:*

(i) If $\int_a^\infty H_r(s)\,\mathrm{d}s = \infty$, *then* (5.42) *with* H_r *instead of* H *holds and* $y(t) \to 0$ *as* $t \to \infty$.

(ii) If $\int_a^\infty H(s)\,\mathrm{d}s < \infty$, *then* (5.43) *with* H_r *instead of* H *holds and* $y(t) \to y(\infty) \in (0,\infty)$ *as* $t \to \infty$.

Proof. We give again only a concise proof. Take $y \in \mathcal{DS}_r \cap \mathcal{SV}$. Then $(rG(y'))' \in \mathcal{RV}(\beta)$. Hence, $-rG(y') \in \mathcal{RV}(\beta+1)$, as so $-y' \in \mathcal{RV}(-1)$, which implies $-y \in \Pi(-ty')$ by (5.44). If $\tilde{L} = L_pF(y)$, then $\tilde{L} \in \mathcal{SV}$ and we have

$$\begin{aligned}-r(t)y'(t)L_G\left(\frac{1}{t}\right) &\sim -r(t)G(y'(t)) = \int_t^\infty p(s)F(y(s))\,\mathrm{d}s = \int_t^\infty s^\beta \tilde{L}(s)\,\mathrm{d}s \\ &\sim \frac{t^{\beta+1}}{-(\beta+1)}\tilde{L}(t) = \frac{t}{-(\beta+1)}p(t)F(y(t))\end{aligned}$$

as $t \to \infty$, where we applied the Karamata theorem. Asymptotic formulae then follow similarly as (5.42) and (5.43) in the proof of Theorem 5.13. □

Remark 5.11. If $p \in \mathcal{RV}(\beta)$, $\beta < -1$, and $r \in \mathcal{RV}(\beta+2)$, then (5.55) holds provided $L_p(t)/L_r(t) \to 0$ as $t \to \infty$. Indeed, by the Karamata theorem we have,

$$\frac{t}{r(t)} \int_t^\infty p(s)\,\mathrm{d}s \sim \frac{tL_p(t)t^{\beta+1}}{-(\beta+1)t^{\beta+2}L_r(t)} = \frac{L_p(t)}{-(\beta+1)L_r(t)}$$

as $t \to \infty$, and the claim follows.

Remark 5.12. Assume that $p \in \mathcal{RV}(\beta)$ and $r \in \mathcal{RV}(\beta+2)$ with $\beta > -1$. Recall that in the previous theorem we assumed $\beta < -1$. Take $y \in \mathcal{DS}_r \cap \mathcal{SV}$. Then we get $\int_a^\infty p(s)F(y(s))\,\mathrm{d}s = \infty$ since the index of regular variation of $pF(y)$ is bigger than -1. Integrating (5.54) from t_0 to t, where t_0 is such that $y(t) > 0, y'(t) < 0$, $t \geq t_0$, we obtain $r(t)G(y'(t)) = r(t_0)G(y'(t_0)) + \int_{t_0}^t p(s)F(y(s))\,\mathrm{d}s$. Hence, if we let t tend to ∞, then $r(t)G(y'(t))$ tends to ∞. Thus y' is eventually positive, which contradicts $y \in \mathcal{DS}_r$. In other words, this observation indicates that $\mathcal{SV}$ solutions should not be searched among $\mathcal{DS}_r$ solutions in this setting. We conjecture that we should take an increasing solution in order to remain in the set $\mathcal{SV}$. Of course, some logical adjustments then have to be made, like taking $\mathcal{RV}$ instead of $\mathcal{RV}_0$ in (5.35). As for $\mathcal{DS}_r$, we conjecture that this class somehow corresponds to $\mathcal{RV}(-1)$ solutions. Note that the integral $\int_a^\infty 1/r(s)\,\mathrm{d}s$ (which is "close" to the integral $\int_a^\infty G^{-1}(M/r(s))\,\mathrm{d}s$) is divergent for $\beta < -1$ resp. convergent for $\beta > -1$ since $1/r \in \mathcal{RV}(-\beta-2)$.

We have not mentioned the remaining possibility so far, namely $\beta = -1$. This border case is probably the most difficult one, and surely will require a quite different approach. The direct use of the Karamata theorem is problematic in contrast to the corresponding situations in other cases. If $p \in \mathcal{RV}(-1)$ and $r \in \mathcal{RV}(1)$, we cannot even say whether $\int_a^\infty p(s)\,\mathrm{d}s$, $\int_a^\infty 1/r(s)\,\mathrm{d}s$ are convergent or divergent. In fact, the situation is more tangled because of presence of nonlinearities F, G, where $\mathcal{SV}$ components L_F, L_G are supposed to have a stronger effect than in the case $\beta \neq -1$.

As a conclusion of this section, we indicate some further directions for a possible future research. Asymptotic theory of nearly linear equations offers many interesting questions. This section contains some answers but there are many issues which could be followed further. There is also some space for improving the presented results. We conjecture that the results can be generalized in the sense of replacing condition (5.35) by $F(|\cdot|), G(|\cdot|) \in \mathcal{RV}_0(\gamma)$, $\gamma > 0$, which would lead to a "nearly half-linear equation." It is expected that — within our setting, with taking $\mathcal{RV}$ instead of $\mathcal{RV}_0$ in (5.35) — increasing solutions of (5.34) are in $\mathcal{RV}(1)$ and asymptotic formulae can be established. In contrast to the linear case, a reduction of order formula is not at disposal. A topic which would also be of interest is to obtain more precise information about $\mathcal{SV}$ solutions of (2.1), for instance, by means of the class $\Pi\mathrm{R}_2$, cf. Theorem 2.13.

Chapter 6

Concluding remarks

It is evident that many other works can be found in which the theory of regularly varying (or of somehow related) functions is applied to study differential equations (or dynamic equations). However, the aim of this text is such that it cannot cover everything (and with details). In the last chapter we briefly mention at least few another works.

6.1 More on differential equations in the framework of regular variation

Appleby in [4] considers the rate of convergence to equilibrium of Volterra integrodifferential equations with infinite memory. It is shown that if the kernel of Volterra operator is regularly varying at infinity, and the initial history is regularly varying at minus infinity, then the rate of convergence to the equilibrium is regularly varying at infinity, and the exact pointwise rate of convergence can be determined in terms of the rate of decay of the kernel and the rate of growth of the initial history. The result is considered both for a linear Volterra integrodifferential equation as well as for the delay logistic equation from population biology.

The concept of subexponential functions (which are somehow related to $\mathcal{RV}$ functions) is used, for instance, by Appleby, Györi, and Reynolds in [5]; the paper examines the asymptotic behavior of solutions of scalar linear integro-differential equations. See also Appleby, Reynolds [8, 7].

Regular variation was used by Marić, Radašin in [109] to study equations arising in boundary-layer theory, see also [105, Chapter 4]. The core of such considerations is based on the work by McLeod [118]. Van den Berg in [15] established a result on the asymptotics of some solutions to a first order nonlinear differential equation; a conspicuous feature of his consideration is its relation to the nonstandard asymptotic analysis.

Several papers exist devoted to investigation of second order equations with deviating argument in the framework of regular variation, see Kusano, Marić [95, 96, 97, 98] for linear case and Kusano, Manojlović, Tanigawa [92] and Tanigawa [161, 163] for half-liner case. Note that the last mentioned paper utilizes the concept of generalized regular variation. Common features of these works are that a crucial role is played by the results for equations without retarded and advanced arguments which can be seen as modifications of some theorems in Chapters 2 and 3. A desired solution is then obtained by the fixed point technique.

6.2 Regularly varying sequences and difference equations

As mentioned in Subsection 1.3.3, the concept of regularly varying sequences was introduced already by Karamata. Much later this theory was used to study asymptotic properties of linear and half-linear difference equations in Matucci and Řehák [119, 120, 121, 122], discrete versions of some of the statements from Chapters 2 and 3 can be revealed there. It should however be emphasized that the discrete case requires to find new ways of the proofs at many points. A somewhat different approach to the use of regularly varying sequences in linear difference equations is represented by Kooman [81], see also related paper by Kooman [80]. Second order Emden-Fowler type difference equations is investigated in the framework of regular variation in [1] by Agarwal and Manojlović. Other approach in the study of Emden-Fowler type difference equations is represented by the papers [76, 77, 78] of Kharkov. Some considerations in Kharkov's papers can be seen as a discrete analogues of the results by Evtukhov et al., see e.g. Subsection 4.3.7.

6.3 Regular variation on time scales and dynamic equations

The concept of regularly varying functions on time scales (or measure chains) is introduced in [141] by Řehák in order to study asymptotic behavior of dynamic equations (which unify and extend differential and difference equations), see Subsection 1.3.3. Other applications to linear and half-linear dynamic equations on time scales can be found in Řehák and Vítovec [152, 153]. The results can be viewed as a unification and extension of some of the statements from Chapters 2 and 3 and corresponding statements for difference equations. Note that an important role is played by an additional condition on the graininess $\mu(t)$; it is somehow necessary to assume that $\mu(t) = o(t)$ as $t \to \infty$.

6.4 q-regular variation and q-difference equations

The concept of q-regularly varying functions was introduced in [151] by Řehák and Vítovec in order to study asymptotic behavior of q-difference equations, see

Subsection 1.3.3. Linear q-difference equations in the framework of this theory are studied in Řehák [142, 144]. Half-linear q-difference equations are studied in [143] Řehák and Vítovec [154]. These results can be understood as a q-version of some of the statements from Chapters 2 and 3. It is worthy of note that a specific approach is used for q-difference equations; this is due to pleasant properties of q-regularly varying functions. Certain generalization of q-regular variation was introduced in Řehák [145] and applied in the study of general linear second order q-difference equations. See also Řehák [148] where the classical Poincaré-Perron type result was applied to examine generalized q-regularly varying solutions of n-th order linear q-difference equations.

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